On doubly structured matrices and pencils that arise in linear response theory

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Abstract

We discuss matrix pencils with a double symmetry structure that arise in the Hartree-Fock model in quantum chemistry. We derive anti-triangular condensed forms from which the eigenvalues can be read off. Ideally these would be condensed forms under unitary equivalence transformations that can be turned into stable (structure preserving) numerical methods. For the pencils under consideration this is a difficult task and not always possible. We present necessary and sufficient conditions when this is possible. If this is not possible then we show how we can include other transformations that allow to reduce the pencil to an almost anti-triangular form.

Keywords: Selfadjoint matrix, skew-adjoint matrix, matrix pencil, Hartree-Fock model, random phase approximation, anti-triangular form, canonical form, condensed form, skew Hamiltonian/Hamiltonian pencil

AMS Subject Classification: 65F15, 15A21, 15A22, 15A57

1 Introduction

In this paper we discuss condensed forms for matrices and pencils with a double symmetry structure that arise in quantum chemistry. The most general formulation of the linear response eigenvalue equation has the form $\lambda \mathcal{E}_0 x = A_0 x$, where $x \in \mathbb{C}^n$ and

$$\lambda \mathcal{E}_0 - A_0 := \lambda \begin{bmatrix} C & Z \\ -Z & -C \end{bmatrix} - \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \tag{1}$$

with $A, B, C, Z \in \mathbb{C}^{n \times n}$, $A = A^*$, $B = B^*$, $C = C^*$, $Z = -Z^*$, see [9, 18].

There are important special cases in which the pencil has extra properties. The simplest response function model is the time-dependent Hartree-Fock model, also called the random phase approximation (RPA). In this special case C is the identity and Z is the zero matrix, see [9, 18]. Then the generalized eigenvalue problem (1) reduces to the problem of finding the eigenvalues of the matrix

$$\mathcal{L}_0 = \begin{bmatrix} A & B \\ -B & -A \end{bmatrix}, \tag{2}$$

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where A, B are as in (1).

For stable Hartree-Fock ground state wave functions, it is furthermore known that $A - B$ and $A + B$ are positive definite and all eigenvalues of $\mathcal{L}_0$ are real [9, 21]. However, also the general case is of interest. In multiconfigurational RPA the matrix $\mathcal{E}_0$ in (1) may be singular, see [9].

The double symmetry structure of the special matrices $\mathcal{E}_0$ and A_0 in (1) and $\mathcal{L}_0$ in (2) can be understood as symmetry with respect to indefinite scalar products. Recall the following well-known definitions, see, e.g., [7, 13].

Definition 1.1 *Let $\mathcal{H} \in \mathbb{C}^{n\times n}$ be nonsingular and Hermitian or skew-Hermitian.*

1. *A matrix $A \in \mathbb{C}^{n\times n}$ is called $\mathcal{H}$-selfadjoint if $A^*\mathcal{H} = \mathcal{H}A$.*
2. *A matrix $S \in \mathbb{C}^{n\times n}$ is called $\mathcal{H}$-skew-adjoint if $S^*\mathcal{H} = -\mathcal{H}S$.*
3. *A matrix $U \in \mathbb{C}^{n\times n}$ is called $\mathcal{H}$-unitary if $U^*\mathcal{H}U = \mathcal{H}$.*

Defining the matrices

$$\Sigma_n = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \quad \Gamma_n = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

(we drop the index n if the size of the matrices is clear from the context), we immediately see that in (1) $\mathcal{E}_0$ is Hermitian and Γ-skew-adjoint, A_0 is Hermitian and Γ-selfadjoint, and $\mathcal{L}_0$ is J-skew-adjoint and Σ-selfadjoint.

In the following, we will rather use the terminology *Hamiltonian*, *skew-Hamiltonian*, and *symplectic* instead of J-skew-adjoint, J-selfadjoint, and J-unitary, respectively, since this is the notation used in much of the literature [17].

It is well known that the set of $\mathcal{H}$-unitary matrices is a Lie group whose corresponding Lie algebra is given by the set of $\mathcal{H}$-skew-adjoint matrices, whereas the set of $\mathcal{H}$-selfadjoint matrices is a Jordan algebra. Furthermore, it is known that the spectrum of $\mathcal{H}$-unitary, $\mathcal{H}$-skew-adjoint, or $\mathcal{H}$-selfadjoint matrices is symmetric with respect to the unit circle, imaginary axis, or real axis, respectively, see, e.g., [7, 13].

In this paper we develop the algebraic background for numerical algorithms that compute the eigenvalues of matrices and pencils of the forms (2) or (1), respectively, continuing the work of [1, 4, 5, 16]. We are interested in obtaining condensed forms from which the eigenvalues can be easily read off. The transformations for the computation of these forms should satisfy two conditions.

On the one hand, we want to preserve the given structures, because numerical methods that use structure preserving transformations will, in particular, preserve the spectral symmetries that are induced by these structures. This guarantees that in finite precision arithmetic rounding errors will not cause the eigenvalues to lose their symmetries. For the matrices and pencils from linear response theory, the two different structures causes different symmetries, namely symmetry with respect to the imaginary axis and simultaneously symmetry with respect to the real axis. Thus, both structures have to be preserved to maintain the full symmetry of the spectrum.

On the other hand, to achieve numerical stability of the method, we are interested in using unitary transformations, i.e., we like to obtain structured versions of the classical Schur or generalized Schur form, see [8].

In [4, 5], a difficulty in computing the eigenvalues of matrices and pencils of the forms (2) or (1) was observed. In [1] this difficulty was explained by the fact that a reduction to a structured Schur form is not always possible, and a reduction method to a condensed form was presented that uses unitary transformations as well as hyperbolic rotations.

However, the method in [1] was only designed for matrices of the form (2) and not for pencils of the form (1). Moreover, this did not answer the question when a structured Schur form exists, since a complete algebraic analysis of doubly structured matrices was not available at that time. This question was recently analyzed in [16], where canonical forms for doubly structured matrices and pencils have been developed in a very general form. With the help of these results, we are now able to complete the theory of condensed forms for the doubly structured matrices and pencils from linear response theory.

The paper is organized as follows. After some preliminary results in Section 2, in Section 3, and Section 4, we will adapt the forms derived in [16] for the doubly structured matrices and pencils in (2) and (1), respectively. In Section 5 we will use these results to develop necessary and sufficient conditions for the existence of structured Schur forms for both the matrix and the pencil case.

Finally, in Section 6 we generalize the constructive reduction method in [1] to the pencil case, by obtaining a condensed form with the help of unitary transformations whenever possible, but also with the help of non-unitary transformations when this is unavoidable.

We use the following notation. $\lfloor x \rfloor$ stands for the largest integer m that satisfies $m \leq x$. $\mathbb{C}^{m\times n}$ is the set of $m\times n$ complex matrices. $\operatorname{diag}(A_1,\ldots,A_n)$ is the block diagonal matrix with diagonal blocks $A_1,\ldots,A_n$ in that order. $A^{-*} := (A^*)^{-1}$. A signature matrix is a diagonal matrix having only the eigenvalue ± 1. By $(\lambda M - N) \in \mathbb{C}^{k\times k}$, we mean that $\lambda M - N$ is a matrix pencil with both $M, N \in \mathbb{C}^{k\times k}$. The eigenvalue ∞ of a pencil is considered to be an eigenvalue that is both real and purely imaginary, using the convention $-\infty := \infty$, $\overline{\infty} := \infty$, and $\infty^2 := \infty$. Moreover, a matrix $U \in \mathbb{C}^{n\times k}$, $k \leq n$, will be called orthonormal if its columns form an orthonormal set of vectors.

2 Preliminaries

To construct the desired condensed forms we can work directly with the pencil (1) and the matrix (2), but it is more convenient to work on slightly transformed pencils or matrices, respectively, that are still doubly structured. This simplifies the discussion and makes the theory more transparent.

Defining the unitary matrices

$$\mathcal{X}_n = \frac{\sqrt{2}}{2}\begin{bmatrix} I_n & I_n \\ -I_n & I_n \end{bmatrix} \quad \text{and} \quad \mathcal{Y}_n = \Sigma_n \mathcal{X}_n = \frac{\sqrt{2}}{2}\begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} \tag{3}$$

we obtain that

$$\lambda\mathcal{E} - \mathcal{A} := \mathcal{Y}_n(\lambda\mathcal{E}_0 - A_0)\mathcal{X}_n = \lambda\begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}, \tag{4}$$

where $E = C - Z$, $G = A + B$, $H = A - B \in \mathbb{C}^{n\times n}$ and, furthermore, $G = G^*, H = H^*$. In the matrix case we use the transformed matrix

$$\mathcal{A} = \mathcal{X}^{-1}\mathcal{L}_0\mathcal{X} = \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}. \tag{5}$$

It it easy to check that $\mathcal{E}$ is Γ-selfadjoint and skew-Hamiltonian, whereas $\mathcal{A}$ is Γ-selfadjoint and Hamiltonian.

Definition 2.1 *A pencil $\lambda\mathcal{M} - \mathcal{N} \in \mathbb{C}^{2n\times 2n}$ is called*

1. Γ-selfadjoint*, if $\mathcal{M}$ and $\mathcal{N}$ are Γ-selfadjoint.*
2. skew-Hamiltonian/Hamiltonian*, if $\mathcal{M}$ is skew-Hamiltonian and $\mathcal{N}$ is Hamiltonian.*

Thus, the pencil $\lambda\mathcal{E} - \mathcal{A}$ is both Γ-selfadjoint and skew-Hamiltonian/Hamiltonian.

In general, to use a similarity transformation that preserves both structures in a matrix that is doubly structured with respect to J and Γ, we have to restrict the transformation matrices to be in

$$\mathbb{G}_{2n} = \{\mathcal{U} \in \mathbb{C}^{2n\times 2n} \mid \mathcal{U}^*\Gamma\mathcal{U} = \Gamma,\ \mathcal{U}^*J\mathcal{U} = J\} = \left\{ \begin{bmatrix} U & 0 \\ 0 & U^{-*} \end{bmatrix} : \det U \neq 0 \right\},$$

i.e., in the intersection of the Lie groups of Γ-unitary and symplectic matrices.

For the pencil case it was shown in [14] that the so-called *J-congruence transformations* preserve the structure of skew-Hamiltonian/Hamiltonian pencils. Analogously, we define Γ-*congruence transformations* that preserve the structure of Γ-selfadjoint pencils.

Definition 2.2 *Let $\lambda A - B, \lambda C - D \in \mathbb{C}^{2n\times 2n}$ be two matrix pencils and let $\mathcal{H}$ be a nonsingular, skew-Hermitian or Hermitian matrix. Then $\lambda A - B$ and $\lambda C - D$ are called $\mathcal{H}$-*congruent *if there exists a nonsingular matrix $P \in \mathbb{C}^{2n\times 2n}$ such that*

$$\lambda C - D = \mathcal{H}^{-1}P^*\mathcal{H}(\lambda A - B)P.$$

It is easy to verify that if P is in the set

$$\begin{aligned} \mathbb{GP}_{2n} &:= \{\mathcal{U} \in \mathbb{C}^{2n\times 2n} \mid J^{-1}\mathcal{U}^*J = \Gamma^{-1}\mathcal{U}^*\Gamma,\ \det\mathcal{U} \neq 0\} \\ &= \left\{ \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} : U_1, U_2 \in \mathbb{C}^{n\times n}, \det(U_1U_2) \neq 0 \right\}, \end{aligned}$$

then the J-congruence transformation

$$(\lambda A - B) \mapsto J^{-1}P^*J(\lambda A - B)P$$

is also a Γ-congruence transformation and preserves the structure of pencils that are doubly structured with respect to J and Γ. For the computation of structured Schur forms, the similarity transformation matrices and the equivalence transformation matrices are restricted to be in the intersections of the group $\mathbb{U}_{2n}$ of unitary matrices and $\mathbb{G}_{2n}$, or $\mathbb{GP}_{2n}$, respectively.

Next, for $\lambda \in \mathbb{C}$ and $r \in \mathbb{N}$ we introduce the following matrices in $\mathbb{C}^{r\times r}$,

$$\mathcal{J}_r(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad F_r := \begin{bmatrix} 0 & & (-1)^0 \\ & \iddots & \\ (-1)^{r-1} & & 0 \end{bmatrix},$$

$$Z_r := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}, \quad D_r = \begin{bmatrix} (-1)^0 & & 0 \\ & \ddots & \\ 0 & & (-1)^{r-1} \end{bmatrix}. \tag{6}$$

Proposition 2.3 *The matrices in (6) satisfy the following basic relations.*

1. $F_r^T = F_r^{-1} = (-1)^{r-1} F_r, \quad Z_r^T = Z_r^{-1} = Z_r, \quad D_r^T = D_r^{-1} = D_r;$

2. $D_r = F_r Z_r = (-1)^{r-1} Z_r F_r;$

3. $\mathcal{J}_r(\lambda)^T F_r = -F_r \mathcal{J}_r(-\lambda), \quad \mathcal{J}_r(\lambda)^T Z_r = Z_r \mathcal{J}_r(\lambda), \quad \mathcal{J}_r(\lambda) D_r = -D_r \mathcal{J}_r(-\lambda).$

Proof. The proof is straightforward. □

3 A canonical form for the matrix case

In this section we will present a canonical form for matrices of the form (5). The invariants of matrices that are structured with respect to an indefinite inner product induced by the nonsingular Hermitian matrix $\mathcal{H}$ are well known, see, e.g., [3, 7, 13]. Those invariants clearly include the eigenvalues and their partial multiplicities (i.e., the sizes of Jordan blocks in the Jordan canonical form of the corresponding matrix). In addition, also parameters $\varepsilon \in \{1, -1\}$ that are associated with real eigenvalues of selfadjoint matrices or with purely imaginary eigenvalues of skew-adjoint matrices, respectively, are invariants. The collection of these parameters is sometimes referred to as the *sign characteristic*, see, e.g., [7, 13]. To highlight that these parameters are related to the matrix $\mathcal{H}$, we will use the term $\mathcal{H}$*-structure indices* instead. A general canonical form for doubly structured matrices was recently obtained in [16]. For our particular problem, we obtain the following result.

Theorem 3.1 *Let $\mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be Γ-selfadjoint and Hamiltonian. Then there exists a nonsingular matrix $\mathcal{W} \in \mathbb{C}^{2n\times 2n}$ such that*

$$\begin{aligned} \mathcal{W}^{-1}\mathcal{A}\mathcal{W} &= \operatorname{diag}(A_1, \ldots, A_k), \\ \mathcal{W}^*\Gamma\mathcal{W} &= \operatorname{diag}(S_1, \ldots, S_k), \\ \mathcal{W}^* J \mathcal{W} &= \operatorname{diag}(T_1, \ldots, T_k), \end{aligned} \tag{7}$$

where the blocks A_j, S_j, and T_j have corresponding sizes and are of one and only one of the following forms:

Type 3.1.1 *Even sized blocks associated with the eigenvalue zero:*

$$A_j = \mathcal{J}_{2p}(0), \quad S_j = \varepsilon Z_{2p}, \quad \text{and} \quad T_j = \delta F_{2p},$$

where $p \in \mathbb{N}$ and $\varepsilon, \delta \in \{1, -1\}$. The Γ-structure index of A_j is ε and the J-structure index is $(-1)^p\delta$;

Type 3.1.2 *Paired odd sized blocks associated with the eigenvalue zero:*

$$A_j = \begin{bmatrix} \mathcal{J}_{2p+1}(0) & 0 \\ 0 & \mathcal{J}_{2p+1}(0) \end{bmatrix}, \; S_j = \begin{bmatrix} 0 & Z_{2p+1} \\ Z_{2p+1} & 0 \end{bmatrix}, \; T_j = \begin{bmatrix} 0 & F_{2p+1} \\ -F_{2p+1} & 0 \end{bmatrix},$$

where $p \in \mathbb{N}$. The Γ-structure indices of the two blocks of A_j are $1, -1$ and the J-structure indices are $1, -1$;

Type 3.1.3 *Blocks associated with a pair $\lambda, -\lambda$ of non-zero real eigenvalues:*

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & -\mathcal{J}_p(\lambda) \end{bmatrix}, \quad S_j = \varepsilon \begin{bmatrix} Z_p & 0 \\ 0 & -Z_p \end{bmatrix}, \quad \text{and} \quad T_j = \begin{bmatrix} 0 & Z_p \\ -Z_p & 0 \end{bmatrix},$$

where $\lambda > 0$*,* $p \in \mathbb{N}$*, and* $\varepsilon \in \{1, -1\}$*. The* Γ*-structure index of* $\mathcal{J}_p(\lambda)$ *is* ε *and the* Γ*-structure index of* $-\mathcal{J}_p(\lambda)$ *is* $(-1)^p\varepsilon$*;*

Type 3.1.4 *Blocks associated with a pair* $i\alpha, -i\alpha$ *of non-zero, purely imaginary eigenvalues:*

$$A_j = \begin{bmatrix} i\mathcal{J}_p(\alpha) & 0 \\ 0 & -i\mathcal{J}_p(\alpha) \end{bmatrix}, \quad S_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad \text{and} \quad T_j = i\delta \begin{bmatrix} -Z_p & 0 \\ 0 & Z_p \end{bmatrix},$$

where $\alpha > 0$*,* $p \in \mathbb{N}$*, and* $\delta \in \{-1, 1\}$*. The* J*-structure index of* $i\mathcal{J}_p(\alpha)$ *is* δ *and the* J*-structure index of* $-i\mathcal{J}_p(\alpha)$ *is* $(-1)^p\delta$*;*

Type 3.1.5 *Blocks associated with a quadruple* $\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda}$ *of non-real, non-purely imaginary eigenvalues:*

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\overline{\lambda}) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\overline{\lambda}) \end{bmatrix},$$

$$S_j = \begin{bmatrix} 0 & 0 & Z_p & 0 \\ 0 & 0 & 0 & Z_p \\ Z_p & 0 & 0 & 0 \\ 0 & Z_p & 0 & 0 \end{bmatrix}, \quad \text{and} \quad T_j = \begin{bmatrix} 0 & 0 & 0 & Z_p \\ 0 & 0 & Z_p & 0 \\ 0 & -Z_p & 0 & 0 \\ -Z_p & 0 & 0 & 0 \end{bmatrix},$$

where $p \in \mathbb{N}$ *and* $\lambda \in \mathbb{C}$ *such that* $Re(\lambda), Im(\lambda) > 0$*.*

Moreover, the form (7) is uniquely determined up to the permutation of blocks.

Proof. The proof follows directly from Theorem 4.10 in [16], considering $\mathcal{A}$ as a doubly structured matrix with respect to the Hermitian matrices Γ and iJ. □

Theorem 3.1 displays all the invariants of a matrix $\mathcal{A}$ that is structured with respect to the indefinite inner products induced by J and Γ. However, the canonical form is now structured with respect to $\mathcal{W}^*J\mathcal{W}$ and $\mathcal{W}^*\Gamma\mathcal{W}$. But for the development of structured numerical algorithms, we will need a canonical form that displays all the invariants and that is still structured with respect to Γ and J. This canonical form is as follows.

Theorem 3.2 *Let* $\mathcal{A} \in \mathbb{C}^{2n\times 2n}$ *be* Γ*-selfadjoint and Hamiltonian. Then there exists a matrix* $\mathcal{U} \in \mathbb{G}_{2n}$ *such that*

$$\mathcal{U}^{-1}\mathcal{A}\mathcal{U} = \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}, \tag{8}$$

with

$$G = \operatorname{diag}(G_1, \ldots, G_k), \;\; H = \operatorname{diag}(H_1, \ldots, H_k),$$

where G_j *and* H_j *have corresponding sizes and are of one and only one of the following forms:*

Type 3.2.1 *Even sized blocks associated with the eigenvalue zero:*

$$\begin{array}{lll} \text{(a)} & G_j = \varepsilon Z_p, & H_j = \varepsilon \mathcal{J}_p(0) Z_p, \;\; \text{or} \\ \text{(b)} & G_j = \varepsilon Z_p \mathcal{J}_p(0), & H_j = \varepsilon Z_p, \end{array}$$

where $p \in \mathbb{N}$ *and* $\varepsilon \in \{1, -1\}$*;*

Type 3.2.2 *Paired odd sized blocks associated with the eigenvalue zero:*

$$G_j = Z_{2p+1}\mathcal{J}_{2p+1}(0) \quad \text{and} \quad H_j = \mathcal{J}_{2p+1}(0) Z_{2p+1},$$

where $p \in \mathbb{N}$;

Type 3.2.3 *Blocks associated with a pair $\lambda, -\lambda$ of non-zero real eigenvalues:*

$$G_j = \varepsilon Z_p \mathcal{J}_p(\lambda) \quad \textit{and} \quad H_j = \varepsilon \mathcal{J}_p(\lambda) Z_p,$$

where $\lambda > 0$, $p \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$;

Type 3.2.4 *Blocks associated with a pair $i\alpha$, $-i\alpha$ of non-zero purely imaginary eigenvalues:*

$$G_j = -\delta Z_p \mathcal{J}_p(\lambda) \quad \textit{and} \quad H_j = \delta \mathcal{J}_p(\lambda) Z_p,$$

where $\alpha > 0$, $p \in \mathbb{N}$, $\delta \in \{1, -1\}$;

Type 3.2.5 *Blocks associated with a quadruple $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ of non-real, non-purely imaginary eigenvalues:*

$$G_j = \begin{bmatrix} 0 & Z_p \mathcal{J}_p(\overline{\lambda}) \\ Z_p \mathcal{J}_p(\lambda) & 0 \end{bmatrix} \quad \textit{and} \quad H_j = \begin{bmatrix} 0 & \mathcal{J}_p(\lambda) Z_p \\ \mathcal{J}_p(\overline{\lambda}) Z_p & 0 \end{bmatrix},$$

where $\mathrm{Re}(\lambda)$, $\mathrm{Im}(\lambda) > 0$, *and $p \in \mathbb{N}$.*

Moreover, the form (8) is uniquely determined up to the permutation of blocks.

Proof. By Theorem 3.1, we know that there exists a nonsingular matrix $\mathcal{W} \in \mathbb{C}^{2n \times 2n}$ such that

$$\begin{aligned} \mathcal{W}^{-1} \mathcal{A} \mathcal{W} &= \operatorname{diag}(A_1, \ldots, A_k), \\ \mathcal{W}^* \Gamma \mathcal{W} &= \operatorname{diag}(S_1, \ldots, S_k), \\ \mathcal{W}^* J \mathcal{W} &= \operatorname{diag}(T_1, \ldots, T_k), \end{aligned}$$

where A_j, S_j and T_j are of one of the types of blocks listed in Theorem 3.1.

To these types of blocks we apply simple transformations with matrices P_j that bring A_j, S_j, and T_j to the forms

$$P_j^{-1} A_j P_j = \begin{bmatrix} 0 & G_j \\ H_j & 0 \end{bmatrix}, \quad \Gamma_{q_j} = P_j^* S_j P_j, \quad \text{and} \quad J_{q_j} = P_j^* T_j P_j,$$

where $2q_j$ is the size of A_j and G_j, H_j are as asserted. Then, taking the product $\mathcal{W} \cdot \operatorname{diag}(P_1, \ldots, P_k)$ and multiplying from the right with an appropriate block permutation matrix produces a matrix $\mathcal{U}$ satisfying

$$\mathcal{U}^* \Gamma \mathcal{U} = \Gamma \quad \text{and} \quad \mathcal{U}^* J \mathcal{U} = J,$$

i.e. $\mathcal{U} \in \mathbb{G}_{2n}$, such that $\mathcal{U}^{-1} \mathcal{A} \mathcal{U}$ has the desired form.

In the following, we explicitely give the transformation matrix P_j that transforms the blocks of type 3.1.x in Theorem 3.1 to the corresponding blocks of type 3.2.x in Theorem 3.2, where we use the same symbols for the parameters as in Theorem 3.1.

Type 3.2.1 If the triple (A_j, S_j, T_j) is of Type 3.1.1 of Theorem 3.1, then we have to distinguish two cases. In the case $\varepsilon\delta = 1$, the transformation matrix P_j is of the form

$$P_j = [\varepsilon e_{2p-1}, \varepsilon e_{2p-3}, \ldots, \varepsilon e_1, e_2, e_4, \ldots, e_{2p}],$$

where e_j denotes the j-th unit vector of dimension $2n$. In the case $\varepsilon\delta = -1$, the transformation matrix P_j is of the form

$$P_j = [\varepsilon e_{2p}, \varepsilon e_{2p-2}, \ldots, \varepsilon e_2, e_1, e_3, \ldots, e_{2p-1}].$$

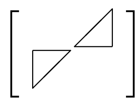
Then G_j and H_j are as in Type 3.2.1 (a) if $\varepsilon\delta = 1$, or as in Type 3.2.1 (b) if $\varepsilon\delta = -1$;

Type 3.2.2 $P_j = \frac{1}{2}\begin{bmatrix} Z_{2p+1} + F_{2p+1} & I_{2p+1} - D_{2p+1} \\ Z_{2p+1} - F_{2p+1} & I_{2p+1} + D_{2p+1} \end{bmatrix}$;

Type 3.2.3 $P_j = \frac{\sqrt{2}}{2}\begin{bmatrix} Z_p & \varepsilon I_p \\ -\varepsilon Z_p & I_p \end{bmatrix}$;

Type 3.2.4 $P_j = \frac{\sqrt{2}}{2}\begin{bmatrix} Z_p & i\delta I_p \\ i\delta Z_p & I_p \end{bmatrix}$;

Type 3.2.5 $P_j = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & Z_p & I_p & 0 \\ 0 & Z_p & -I_p & 0 \\ Z_p & 0 & 0 & I_p \\ -Z_p & 0 & 0 & I_p \end{bmatrix}$. $\square$

Remark 3.3 Note that all submatrices $\begin{bmatrix} 0 & G_j \\ H_j & 0 \end{bmatrix}$ of (8) have the pattern .

We have seen in this section that matrices that are Γ-selfadjoint and Hamiltonian can be transformed to a structured canonical form that is the analogue of the classical Jordan canonical form. In the next section we derive similar canonical forms for the corresponding doubly structured pencils.

4 Canonical forms for the pencil case

In this section, we discuss canonical forms for regular pencils $\lambda\mathcal{E} - \mathcal{A}$ in the form (4). Recall that a pencil $\lambda\mathcal{E} - \mathcal{A}$ is called *regular* if and only if $\det(\lambda\mathcal{E} - \mathcal{A})$ is not identically zero. To do this, we first split the pencil into two parts corresponding to finite and infinite eigenvalues, respectively.

Theorem 4.1 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be a regular, Γ-selfadjoint, and skew-Hamiltonian/-Hamiltonian pencil. Then there exist nonsingular matrices $\mathcal{W}_1, \mathcal{W}_2 \in \mathbb{C}^{2n\times 2n}$ such that*

$$\mathcal{W}_2^{-1}(\lambda\mathcal{E} - \mathcal{A})\mathcal{W}_1 = \lambda\begin{bmatrix} I_{2k} & 0 \\ 0 & \mathcal{E}_\infty \end{bmatrix} - \begin{bmatrix} A_f & 0 \\ 0 & I_{2m} \end{bmatrix},$$

$$\mathcal{W}_1^*\Gamma\mathcal{W}_2 = \begin{bmatrix} \Gamma_k & 0 \\ 0 & S_\infty \end{bmatrix}, \quad \mathcal{W}_1^* J \mathcal{W}_2 = \begin{bmatrix} J_k & 0 \\ 0 & T_\infty \end{bmatrix},$$

where $A_f \in \mathbb{C}^{2k\times 2k}$ is Γ_k-selfadjoint and Hamiltonian and $\mathcal{E}_\infty$ is S_∞-selfadjoint and T_∞-skew-adjoint. Furthermore, we have

$$\mathcal{E}_\infty = \operatorname{diag}(E_1, \ldots, E_l), \quad S_\infty = \operatorname{diag}(S_1, \ldots, S_l), \quad T_\infty = \operatorname{diag}(T_1, \ldots, T_l), \tag{9}$$

where the blocks E_j, S_j, and T_j have corresponding sizes and are of one and only one of the following forms:

Type 4.1.1 *Paired even sized blocks associated with the eigenvalue* ∞*:*

$$E_j = \begin{bmatrix} \mathcal{J}_{2p}(0) & 0 \\ 0 & \mathcal{J}_{2p}(0) \end{bmatrix}, \quad S_j = \begin{bmatrix} 0 & Z_{2p} \\ Z_{2p} & 0 \end{bmatrix}, \quad \text{and} \quad T_j = \begin{bmatrix} 0 & F_{2p} \\ -F_{2p} & 0 \end{bmatrix},$$

where $p \in \mathbb{N}$*;*

Type 4.1.2 *Odd sized blocks associated with the eigenvalue* ∞*:*

$$E_j = \mathcal{J}_{2p+1}(0), \quad S_j = \varepsilon Z_{2p+1}, \quad \text{and} \quad T_j = \delta F_{2p+1},$$

where $\varepsilon, \delta \in \{1, -1\}$*,* $p \in \mathbb{N}$*. The* Γ*-structure index of* E_j *is* ε *and the* J*-structure index is* $(-1)^p\delta$*.*

Proof. By Theorem 5.1 in [16], there exist nonsingular matrices $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{C}^{2n\times 2n}$ such that

$$\mathcal{Z}_1^{-1}(\lambda\mathcal{E} - \mathcal{A})\mathcal{Z}_2 = \lambda \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} M & 0 \\ 0 & I_r \end{bmatrix},$$

$$\mathcal{Z}_2^*\Gamma\mathcal{Z}_1 = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad \mathcal{Z}_2^*J\mathcal{Z}_1 = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix},$$

where N is nilpotent, Q_1, Q_2, R_2 are Hermitian, R_1 is skew-Hermitian, and the following identities hold:

$$\begin{aligned} M^*R_1 &= -R_1M, \quad M^*Q_1 = Q_1M, \\ N^*R_2 &= -R_2N, \quad N^*Q_2 = Q_2N. \end{aligned} \tag{10}$$

(Note that although J is skew-Hermitian and $\mathcal{E}$ was skew-Hamiltonian, i.e., J-selfadjoint, R_2 is now Hermitian and N is R_2-skew-adjoint.) Since the pencil $\lambda\Gamma - J$ only has the eigenvalues $1, -1$ with partial multiplicities equal to one, the same holds for the pencil $\lambda Q_1 - R_1$. Equivalently, the Hermitian pencil $\lambda Q_1 - iR_1$ has only the eigenvalues $i, -i$ with partial multiplicities equal to one. But, since non-real eigenvalues of Hermitian pencils always occur in pairs, see [20], it follows that the algebraic multiplicities of i and $-i$ are equal, say k. But then it follows from the well-known results on canonical forms of Hermitian pencils [20], that there exists a nonsingular matrix $\mathcal{V}$ such that

$$\mathcal{V}^*(\lambda Q_1 - iR_1)\mathcal{V} = \lambda\Gamma_k - iJ_k.$$

Moreover, since N is nilpotent and by (10) it is also R_2-skew-adjoint and Q_2-selfadjoint, and since the Hermitian pencil $\lambda R_2 - Q_2$ only has eigenvalues $1, -1$ with partial multiplicities equal to one, it follows from Theorem 4.10 in [16] that there exists a nonsingular matrix $\mathcal{U}$ such that $\mathcal{U}^{-1}N\mathcal{U} = \mathcal{E}_\infty$, $\mathcal{U}^*Q_2\mathcal{U} = S_\infty$, and $\mathcal{U}^*R_2\mathcal{U} = T_\infty$, where $\mathcal{E}_\infty$, S_∞, and T_∞ are as in (9). Setting

$$\mathcal{W}_1 = \mathcal{Z}_2 \begin{bmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{U} \end{bmatrix}, \quad \mathcal{W}_2 = \mathcal{Z}_1 \begin{bmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{U} \end{bmatrix}$$

then gives the desired result, since $A_f := \mathcal{V}^{-1}M\mathcal{V}$ is Γ_k-selfadjoint and Hamiltonian. □

We immediately obtain the following corollary of Theorem 4.1 and Theorem 3.1.

Corollary 4.2 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be a regular, Γ-selfadjoint, and skew-Hamiltonian/-Hamiltonian pencil. Then there exist nonsingular matrices $\mathcal{W}_1, \mathcal{W}_2$ such that*

$$\mathcal{W}_2^{-1}(\lambda\mathcal{E} - \mathcal{A})\mathcal{W}_1 = \lambda \begin{bmatrix} I_{2k} & 0 \\ 0 & \mathcal{E}_\infty \end{bmatrix} - \begin{bmatrix} A_f & 0 \\ 0 & I_{2m} \end{bmatrix}, \tag{11}$$

$$\mathcal{W}_1^*\Gamma\mathcal{W}_2 = \begin{bmatrix} S_f & 0 \\ 0 & S_\infty \end{bmatrix}, \quad \mathcal{W}_1^*J\mathcal{W}_2 = \begin{bmatrix} T_f & 0 \\ 0 & T_\infty \end{bmatrix},$$

where $A_f, S_f, T_f \in \mathbb{C}^{2k\times 2k}$ are in the canonical form (8), and $\mathcal{E}_\infty, S_\infty, T_\infty \in \mathbb{C}^{(2n-2k)\times(2n-2k)}$ are as in (9).

As in the matrix case, we would prefer a simple form that displays the eigenvalues and that still is Γ-selfadjoint and skew-Hamiltonian/Hamiltonian. However, this task is not as easy as in the matrix case. The problem in the pencil case is that in the canonical form (11) odd-sized blocks associated with the eigenvalue ∞ need not occur in pairs. Consider the following example.

Example 4.3 The pencil $\lambda\mathcal{E} - \mathcal{A}$, where

$$\mathcal{E} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \mathcal{A} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right],$$

is regular, Γ-selfadjoint, and skew-Hamiltonian/Hamiltonian. Setting

$$\mathcal{W}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{W}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

we obtain the canonical form

$$\mathcal{W}_2^{-1}\mathcal{E}\mathcal{W}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{W}_2^{-1}\mathcal{A}\mathcal{W}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{W}_1^*\Gamma\mathcal{W}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{W}_1^*J\mathcal{W}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Thus, the pencil $\lambda\mathcal{E} - \mathcal{A}$ has two Jordan blocks associated with ∞. The first one is of size three with parameters $\varepsilon_1 = 1$ and $\delta_1 = 1$ as in Theorem 4.1 (hence, the Γ-structure index is 1 and the J-structure index is -1) and the second one is of size one with parameters $\varepsilon_2 = 1$ and $\delta_2 = -1$ as in Theorem 4.1 (hence, the Γ-structure index is 1 and the J-structure index is -1).

Example 4.3 shows the difficulties that are caused by the lack of pairing of odd-sized blocks associated with the eigenvalue ∞. It is difficult to find a simple form that nicely displays the Kronecker structure of $\lambda\mathcal{E}-\mathcal{A}$ if we want to keep the two-by-two block structure of $\mathcal{E}$. In Appendix 1, for completeness, we present such a form without the technical proof. Here, we restrict ourselves to the case that the odd-sized blocks associated with the eigenvalue ∞ occur in pairs in the following sense.

Definition 4.4 *Let $\lambda\mathcal{E}-\mathcal{A}\in\mathbb{C}^{2n\times 2n}$ be a regular, Γ-selfadjoint, and skew-Hamiltonian/-Hamiltonian pencil, and let $n(\infty,k,\eta)$ denote the number of Jordan blocks associated with ∞ in the canonical form (11) that have size k, and for that the corresponding structure indices δ and ε in (11) satisfy $\delta\varepsilon=\eta$. Then $\lambda\mathcal{E}-\mathcal{A}$ is called* ∞-regular *if for any odd $k\in\mathbb{N}$ we have that*

$$n(\infty,k,1)=n(\infty,k,-1).$$

Thus, for an ∞-regular pencil, the odd-sized blocks associated with ∞ have to be paired with respect to the sign of the product of their structure indices. At first glance, this condition sounds rather special and hard to check. However, it turns out that this condition is satisfied if the pencil is of *differential-index at most one*, i.e., all partial multiplicities associated with the eigenvalue ∞ are less or equal to one. This is an important case in many applications that can be achieved via an index reduction process [2, 10, 11, 12].

Proposition 4.5 *Let $\lambda\mathcal{E}-\mathcal{A}\in\mathbb{C}^{2n\times 2n}$ be a regular, Γ-selfadjoint, skew-Hamiltonian/Hamiltonian pencil of differential index at most one. Then $\lambda\mathcal{E}-\mathcal{A}$ is ∞-regular.*

Proof. Since all partial multiplicities of the eigenvalue ∞ are at most one, it follows from (11) that there exists nonsingular matrices $\mathcal{W}_1$, $\mathcal{W}_2$ such that

$$\mathcal{W}_2^{-1}\mathcal{E}\mathcal{W}_1=\operatorname{diag}(I_{2k},0,0,0,0),\quad \mathcal{W}_2^{-1}\mathcal{A}\mathcal{W}_1=\operatorname{diag}(A,I_p,I_q,I_r,I_s),$$

$$\mathcal{W}_1^*\Gamma\mathcal{W}_2=\operatorname{diag}(\Gamma_k,I_p,I_q,-I_r,-I_s),\quad \mathcal{W}_1^*J\mathcal{W}_2=\operatorname{diag}(J_k,I_p,-I_q,I_r,-I_s),$$

for some $p,q,r,s\in\mathbb{N}$. Since the pencil $\lambda\Gamma-J$ has the eigenvalues $1,-1$ each with multiplicity n, the same still holds for $\mathcal{W}_1^*(\lambda\Gamma-J)\mathcal{W}_2$. This implies $p+s=r+q$. But noting that $p+s$ ($r+q$, respectively) is the number of blocks for that the product of structure-indices is 1 (-1, respectively), it follows that the pencil is ∞-regular. $\square$

For the case of ∞-regular pencils we then have the following structured canonical form.

Theorem 4.6 *Let $\lambda\mathcal{E}-\mathcal{A}\in\mathbb{C}^{2n\times 2n}$ be a ∞-regular, Γ-selfadjoint, and skew-Hamiltonian/-Hamiltonian pencil. Then there exists a nonsingular matrix $\mathcal{W}\in\mathbb{GP}_{2n}$ such that*

$$\begin{aligned}(\Gamma^{-1}\mathcal{W}^*\Gamma)(\lambda\mathcal{E}-\mathcal{A})\mathcal{W} &= (J^{-1}\mathcal{W}^*J)(\lambda\mathcal{E}-\mathcal{A})\mathcal{W}\\ &= \lambda\begin{bmatrix} I_{n_f} & 0 & 0 & 0\\ 0 & E_\infty & 0 & 0\\ 0 & 0 & I_{n_f} & 0\\ 0 & 0 & 0 & E_\infty^*\end{bmatrix}-\begin{bmatrix} 0 & 0 & G_f & 0\\ 0 & 0 & 0 & G_\infty\\ H_f & 0 & 0 & 0\\ 0 & H_\infty & 0 & 0\end{bmatrix},\end{aligned}$$

where G_f and H_f are in the canonical form (8) of Theorem 3.2. Furthermore, we have

$$E_\infty=\operatorname{diag}(E_1,\dots,E_k),\quad G_\infty=\operatorname{diag}(G_1,\dots,G_k),\quad H_\infty=\operatorname{diag}(H_1,\dots,H_k),$$

where the blocks E_j, G_j, and H_j have corresponding sizes and are of one and only one of the following forms:

Type 4.6.1 *Paired even sized blocks associated with the eigenvalue ∞:*

$$E_j = Z_{2p}\mathcal{J}_{2p}(0) \quad \text{and} \quad G_j = H_j = Z_{2p},$$

where $p \in \mathbb{N}$;

Type 4.6.2 *Two odd sized blocks associated with the eigenvalue ∞:*

$$E_j = \varepsilon Z_{2p+1}\mathcal{J}_{2p+1}(0) \quad \text{and} \quad G_j = H_j = \varepsilon Z_{2p+1},$$

where $p \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$.

Proof. In view of Theorems 4.1 and 3.2, it is sufficient to consider the case that $\mathcal{E}$ is nilpotent. But then, by Theorem 4.1, there exist nonsingular matrices $\mathcal{W}_1$ and $\mathcal{W}_2$ such that

$$\begin{array}{rclrcl} \mathcal{W}_2^{-1}\mathcal{E}\mathcal{W}_1 & = & \operatorname{diag}(E_1, \ldots, E_l), & \mathcal{W}_1^*\Gamma\mathcal{W}_2 & = & \operatorname{diag}(S_1, \ldots, S_l), \\ \mathcal{W}_2^{-1}\mathcal{A}\mathcal{W}_1 & = & I_{2n}, & \mathcal{W}_1^* J\mathcal{W}_2 & = & \operatorname{diag}(T_1, \ldots, T_l), \end{array}$$

where E_j, S_j, T_j are of one of the types of Theorem 4.1. We consider these types separately:

Type 4.6.1 If (E_j, S_j, T_j) is of Type 4.1.1 with parameter p as in Theorem 4.1, then setting

$$P_j = \frac{1}{2}\begin{bmatrix} I_{2p} + D_{2p} & I_{2p} - D_{2p} \\ I_{2p} - D_{2p} & I_{2p} + D_{2p} \end{bmatrix} \quad \text{and} \quad Q_j = \frac{1}{2}\begin{bmatrix} Z_{2p} - F_{2p} & Z_{2p} + F_{2p} \\ Z_{2p} + F_{2p} & Z_{2p} - F_{2p} \end{bmatrix},$$

we obtain that

$$Q_j^{-1}E_jP_j = \begin{bmatrix} Z_{2p}\mathcal{J}_{2p}(0) & 0 \\ 0 & \mathcal{J}_{2p}(0)^* Z_{2p} \end{bmatrix}, \quad Q_j^{-1}A_jP_j = \begin{bmatrix} 0 & Z_{2p} \\ Z_{2p} & 0 \end{bmatrix},$$

$$P_j^* S_j Q_j = \Gamma_{2p}, \quad \text{and} \quad P_j^* T_j Q_j = J_{2p}.$$

Type 4.6.2 Let (E_j, S_j, T_j) be of Type 4.1.2 with parameters p, ε, δ as in Theorem 4.1. Since the pencil is ∞-regular, we know that there exists a second triple (E_m, S_m, T_m) with parameters p, $\tilde{\varepsilon}$, $\tilde{\delta}$, where $\varepsilon\delta = -\tilde{\varepsilon}\tilde{\delta}$. Without loss of generality, we may assume that $\varepsilon\delta = 1$, i.e., $\delta = \varepsilon$ and $\tilde{\delta} = -\tilde{\varepsilon}$. Setting

$$P_j = \varepsilon\frac{1}{2}\begin{bmatrix} I_{2p+1} + D_{2p+1} & I_{2p+1} - D_{2p+1} \\ I_{2p+1} - D_{2p+1} & I_{2p+1} + D_{2p+1} \end{bmatrix}, \quad Q_j = \frac{1}{2}\begin{bmatrix} Z_{2p+1} - F_{2p+1} & Z_{2p+1} + F_{2p+1} \\ Z_{2p+1} + F_{2p+1} & Z_{2p+1} - F_{2p+1} \end{bmatrix},$$

with D_j, F_j, Z_j as in (6), we obtain that

$$Q_j^{-1}\begin{bmatrix} E_j & 0 \\ 0 & E_m \end{bmatrix}P_j = \varepsilon\begin{bmatrix} Z_{2p+1}\mathcal{J}_{2p+1}(0) & 0 \\ 0 & \mathcal{J}_{2p+1}(0)^* Z_{2p+1} \end{bmatrix},$$

$$Q_j^{-1}\begin{bmatrix} I_{2p+1} & 0 \\ 0 & I_{2p+1} \end{bmatrix}P_j = \varepsilon\begin{bmatrix} 0 & Z_{2p+1} \\ Z_{2p+1} & 0 \end{bmatrix},$$

$$P_j^*\begin{bmatrix} S_j & 0 \\ 0 & S_m \end{bmatrix}Q_j = \Gamma_{2p+1}, \quad P_j^*\begin{bmatrix} T_j & 0 \\ 0 & T_m \end{bmatrix}Q_j = J_{2p+1}.$$

Taking the products $\mathcal{W}_1 \cdot \operatorname{diag}(P_1, \ldots, P_k)$ and $\mathcal{W}_2 \cdot \operatorname{diag}(Q_1, \ldots, Q_k)$ and applying an appropriate block permutation, yields matrices $\mathcal{U}$ and $\mathcal{W}$ satisfying $\mathcal{W}^*\Gamma\mathcal{U} = \Gamma$ and $\mathcal{W}^* J\mathcal{U} = J$, or equivalently,

$$\mathcal{U}^{-1} = \Gamma^{-1}\mathcal{W}^*\Gamma \quad \text{and} \quad \mathcal{U}^{-1} = J^{-1}\mathcal{W}^* J$$

such that $\mathcal{U}^{-1}(\lambda\mathcal{E} - \mathcal{A})\mathcal{W}$ has the desired form. In particular, we have $\mathcal{W} \in \mathbb{GP}_{2n}$. $\square$

In this section we have extended the structured canonical form for doubly structured matrices to the case of doubly structured pencils under the assumption that the pencil is ∞-regular. For the statement of the general case see Appendix 1.

The presented canonical forms are the algebraic basis for the construction of numerical methods. However, as is well known [8], in general it is not possible to compute such canonical forms via numerical algorithms. For this reason we are interested in condensed forms under unitary transformations. But such forms do not always exist. In the next section we derive necessary and sufficient conditions, when this is the case.

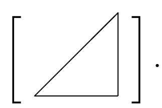
5 Existence of structured Schur forms

In this section, we study structured Schur forms for the doubly structured matrices and pencils under consideration. We begin with the matrix case, i.e.,

$$\mathcal{A} = \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}, \tag{12}$$

where $G, H \in \mathbb{C}^{n\times n}$ are Hermitian. Since the unitary matrices $\mathcal{U}$ in $\mathbb{G}_{2n}$ have the block form $\operatorname{diag}(U, U)$ with U unitary, one has to determine a unitary matrix U such that U^*GU and U^*HU are both in a condensed form from which the eigenvalues of $\mathcal{A}$ can be read off in a simple way. A possible candidate for such a condensed form is that U^*GU and U^*HU are both diagonal. However, it is well known that such a U exists if and only if G and H commute. Hence, such a form exists only for a small set of matrices of the form (12). Another possible candidate is that U^*GU is lower anti-triangular and U^*HU is upper anti-triangular in the following sense.

Definition 5.1 *Let $X = [x_{j,k}] \in \mathbb{C}^{n\times n}$. We say that X is* lower anti-triangular*, if $x_{j,k} = 0$ for $j + k \le n$, i.e., X has the pattern*



Analogously, we say that X is upper anti-triangular *if $x_{j,k} = 0$ for $j+k > n+1$. Moreover, we say that a matrix $\mathcal{A}$ of the form (12) is in* anti-triangular form*, if G is lower anti-triangular and H is upper anti-triangular.*

Anti-triangular Hermitian pencils have been studied in [15], where it was shown that these forms are the natural generalization of the Hamiltonian Schur form, see [17, 19] to the case of Hermitian pencils. Note that Hermitian pencils are related to Hamiltonian matrices by the fact that $\lambda iJ - JM$ is a Hermitian pencil, if M is a Hamiltonian matrix.

Note that if $\mathcal{A}$ is in anti-triangular form, then the eigenvalues of $\mathcal{A}$ are displayed by the entries on the main antidiagonal of G and H. This can be easily verified by applying a row and column permutation to $\mathcal{A}$. For example, if $G = [g_{i,j}]$ and $H = [h_{i,j}]$ then for $1 \le k \le \frac{n}{2}$ and $l = n - k + 1$ every 4×4 submatrix

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & g_{k,l} \\ 0 & 0 & \overline{g}_{k,l} & g_{l,l} \\ h_{k,k} & h_{k,l} & 0 & 0 \\ \overline{h}_{k,l} & 0 & 0 & 0 \end{bmatrix},$$

displays a quadruple $\{\lambda_0, -\lambda_0, \overline{\lambda}_0, -\overline{\lambda}_0\}$ of eigenvalues of A_0, where $\lambda_0 = \sqrt{g_{k,l}\overline{h}_{k,l}}$. In the case that n is odd, we find a distinguished pair of eigenvalues $\lambda_0, -\lambda_0$ that is displayed by the entries in the middle of the anti-diagonals of G and H, i.e., by the submatrix

$$\begin{bmatrix} 0 & g_{r,r} \\ h_{r,r} & 0 \end{bmatrix},$$

where $r = \frac{n+1}{2}$ and $\lambda_0 = \sqrt{g_{r,r}h_{r,r}}$. Since $g_{r,r}h_{r,r}$ is real, λ_0 is necessarily real or purely imaginary.

The corresponding anti-triangular form for the case of a regular pencil

$$\lambda\mathcal{E} - \mathcal{A} = \lambda \begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}, \tag{13}$$

where $E, G, H \in \mathbb{C}^{n\times n}$, G, H Hermitian, is such that E, G, and H are all lower anti-triangular. If this is the case and for n even, if $E = [e_{j,k}], G = [g_{j,k}]$ and $H = [h_{j,k}]$, then for $1 \le k \le \frac{n}{2}$ and $l = n - k + 1$ every 4×4 subpencil

$$\lambda \begin{bmatrix} 0 & e_{k,l} & 0 & 0 \\ e_{l,k} & e_{l,l} & 0 & 0 \\ 0 & 0 & 0 & \overline{e}_{l,k} \\ 0 & 0 & \overline{e}_{k,l} & \overline{e}_{l,l} \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & 0 & g_{k,l} \\ 0 & 0 & \overline{g}_{k,l} & g_{l,l} \\ 0 & h_{k,l} & 0 & 0 \\ \overline{h}_{k,l} & h_{l,l} & 0 & 0 \end{bmatrix}, \tag{14}$$

displays a quadruple $\{\lambda_0, -\lambda_0, \overline{\lambda}_0, -\overline{\lambda}_0\}$ of eigenvalues, where $\lambda_0 = \sqrt{\frac{g_{k,l}h_{k,l}}{e_{k,l}\overline{e}_{l,k}}}$ if $e_{k,l}\overline{e}_{l,k} \neq 0$ and $\lambda_0 = \infty$, otherwise. In n is odd and $r = \frac{n+1}{2}$, analogous to the matrix case, there is a distinguished pair of real or purely imaginary eigenvalues $(\lambda_0, -\lambda_0)$, where $\lambda_0 = \infty$ if $e_{r,r} = 0$ or $\lambda_0 = \sqrt{\frac{g_{r,r}h_{r,r}}{e_{r,r}\overline{e}_{r,r}}}$, otherwise.

It remains to discuss the question when the doubly structured matrix or matrix pencil can be transformed to anti-triangular form. To derive necessary and sufficient conditions for the existence of anti-triangular forms it is sufficient to discuss the pencil case, because if $\mathcal{A}$ as in (12) is J, Γ-congruent to a pencil in anti-triangular form, i.e.,

$$J^{-1}P^*J(\lambda I - \mathcal{A})P = \lambda \begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix},$$

where $P \in \mathbb{GP}_{2n}$, E, G, H are lower anti-triangular and E is invertible, then setting $Q = \operatorname{diag}(E^{-1}, I)$ implies that

$$J^{-1}(PQ)^*J(\lambda I - \mathcal{A})PQ = \lambda \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 & G \\ E^{-*}HE^{-1} & 0 \end{bmatrix}$$

and we find that $J^{-1}(PQ)J\mathcal{A}PQ$ is a matrix in anti-triangular form and, since $P \in \mathbb{GP}_{2n}$ and $J^{-1}(PQ)^*JPQ = I_{2n}$ we obtain $PQ \in \mathbb{G}_{2n}$. Note that if P is unitary, then also E and Q are unitary and hence PQ is also unitary.

To generate the structured anti-triangular forms we derive first an eigenvalue reordering method as well as an off anti-diagonal block elimination technique. Consider an $(8 \times 8)-$subpencil

$$\lambda \begin{bmatrix} \tilde{E} & 0 \\ 0 & \tilde{E}^* \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{H} & 0 \end{bmatrix}$$

of (13) given by the submatrices

$$\tilde{E} = \begin{bmatrix} 0 & 0 & 0 & e_{k,l} \\ 0 & 0 & e_{j,m} & e_{j,l} \\ 0 & e_{m,j} & e_{m,m} & e_{m,l} \\ e_{l,k} & e_{l,j} & e_{l,m} & e_{l,l} \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} 0 & 0 & 0 & g_{k,l} \\ 0 & 0 & g_{j,m} & g_{j,l} \\ 0 & \overline{g}_{j,m} & g_{m,m} & g_{m,l} \\ \overline{g}_{k,l} & \overline{g}_{j,l} & \overline{g}_{m,l} & g_{l,l} \end{bmatrix}$$

and

$$\tilde{H} = \begin{bmatrix} 0 & 0 & 0 & h_{k,l} \\ 0 & 0 & h_{j,m} & h_{j,l} \\ 0 & \overline{h}_{j,m} & h_{m,m} & h_{m,l} \\ \overline{h}_{k,l} & \overline{h}_{j,l} & \overline{h}_{m,l} & h_{l,l} \end{bmatrix},$$

where $\frac{n}{2} \geq j > k$, $l = n - k + 1$, and $m = n - j + 1$, such that

$$g_{k,l}h_{k,l}e_{j,m}\overline{e}_{m,j} \neq g_{j,m}h_{j,m}e_{k,l}\overline{e}_{l,k}, \quad g_{k,l}h_{k,l}\overline{e}_{j,m}e_{m,j} \neq \overline{g}_{j,m}\overline{h}_{j,m}e_{k,l}\overline{e}_{l,k}, \tag{15}$$

i.e., the (8×8)-subpencil displays two disjoint quadruples of eigenvalues $\{\lambda_0, -\lambda_0, \overline{\lambda}_0, -\overline{\lambda}_0\}$ and $\{\mu_0, -\mu_0, \bar{\mu}_0, -\bar{\mu}_0\}$, where $\lambda_0 = \sqrt{\frac{g_{k,l}h_{k,l}}{e_{k,l}\overline{e}_{l,k}}}$ and $\mu_0 = \sqrt{\frac{g_{j,m}h_{j,m}}{e_{j,m}\overline{e}_{m,j}}}$, and $\lambda_0 \neq \pm\mu_0, \pm\overline{\mu}_0$. We want to eliminate the elements $e_{j,l}, e_{l,j}, g_{j,l}$, and $h_{j,l}$ via a transformation with matrices

$$P = \begin{bmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & w & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Forming the product

$$Q^*\tilde{E}P = \begin{bmatrix} 0 & 0 & 0 & e_{k,l} \\ 0 & 0 & e_{j,m} & e_{j,l} + \overline{w}e_{k,l} + ye_{j,m} \\ 0 & e_{m,j} & * & * \\ e_{l,k} & e_{l,j} + \overline{z}e_{m,j} + xe_{l,k} & * & * \end{bmatrix},$$

to eliminate the elements $e_{j,l}$ and $e_{l,j}$, we have to choose w, x, y, z such that the equations

$$e_{j,l} + \overline{w}e_{k,l} + ye_{j,m} = 0 \quad \text{and} \quad e_{l,j} + \overline{z}e_{m,j} + xe_{l,k} = 0$$

hold. The analogous argument for $Q^*\tilde{G}Q$ and $P^*\tilde{H}P$ yields the two additional equations

$$g_{j,l} + \overline{w}g_{k,l} + zg_{j,m} = 0 \quad \text{and} \quad h_{j,l} + \overline{x}h_{k,l} + yh_{j,m} = 0.$$

Altogether, we obtain a linear system in the variables $\overline{w}, \overline{x}, y$ and z given by

$$\begin{bmatrix} \overline{e}_{l,k} & \overline{e}_{m,j} & 0 & 0 \\ h_{k,l} & 0 & h_{j,m} & 0 \\ 0 & g_{j,m} & 0 & g_{k,l} \\ 0 & 0 & e_{j,m} & e_{k,l} \end{bmatrix} \begin{bmatrix} \overline{x} \\ z \\ y \\ \overline{w} \end{bmatrix} = - \begin{bmatrix} \overline{e}_{l,j} \\ h_{j,l} \\ g_{j,l} \\ e_{j,l} \end{bmatrix}.$$

Since the determinant of the system matrix is $-\overline{e}_{l,k}h_{j,m}g_{j,m}e_{k,l} + \overline{e}_{m,j}g_{k,l}h_{k,l}e_{j,m}$, and this term is non-zero by the first condition of (15), we have a unique solution.

In a similar way the second condition of (15) implies that the elements $e_{m,l}, e_{l,m}, g_{m,l}$ and $h_{m,l}$ can be eliminated.

Similarly, in the case that n is odd, $r = \frac{n+1}{2}$, $k < j$, $l = n - k + 1$, and

$$\tilde{E} = \begin{bmatrix} 0 & 0 & e_{k,l} \\ 0 & e_{r,r} & e_{r,l} \\ e_{l,k} & e_{l,r} & e_{l,l} \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} 0 & 0 & g_{k,l} \\ 0 & g_{r,r} & g_{r,l} \\ \overline{g}_{k,l} & \overline{g}_{r,l} & g_{l,l} \end{bmatrix} \quad \text{and} \quad \tilde{H} = \begin{bmatrix} 0 & 0 & h_{k,l} \\ 0 & h_{r,r} & h_{r,l} \\ \overline{h}_{k,l} & \overline{h}_{r,l} & h_{l,l} \end{bmatrix},$$

the condition

$$g_{r,r}h_{r,r}e_{k,l}\overline{e}_{l,k} \neq g_{k,l}h_{k,l}|e_{r,r}|^2, \tag{16}$$

implies that the eigenvalue quadruple $\{\lambda_0, -\lambda_0, \overline{\lambda}_0, -\overline{\lambda}_0\}$ and the pair $\{\mu_0, -\mu_0,\}$ are disjoint, where $\lambda_0 = \sqrt{\frac{g_{k,l}h_{k,l}}{e_{k,l}\overline{e}_{l,k}}}$ and $\mu_0 = \sqrt{\frac{g_{r,r}h_{r,r}}{e_{r,r}\overline{e}_{r,r}}}$. In this case one can eliminate the elements $e_{r,l}, e_{l,r}, g_{r,l}$, and $h_{r,l}$.

Using this elimination technique and applying some permutations to combine blocks that display the same quadruple of eigenvalues to a larger block, we obtain the following proposition.

Proposition 5.2 *Let $(\lambda\mathcal{E} - \mathcal{A}) \in \mathbb{C}^{2n\times 2n}$ be an ∞-regular, Γ-selfadjoint, and skew-Hamiltonian/Hamiltonian pencil such that E, G and H in (13) are lower anti-triangular. Then $\lambda\mathcal{E} - \mathcal{A}$ is Γ, J-congruent to a pencil*

$$\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \lambda \begin{bmatrix} \hat{E} & 0 \\ 0 & \hat{E}^* \end{bmatrix} - \begin{bmatrix} 0 & \hat{G} \\ \hat{H} & 0 \end{bmatrix},$$

where all three matrices $\hat{E}$, $\hat{G}$, $\hat{H}$ are block anti-triangular of the form

$$\hat{X} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 & X_{1,k} \\ \vdots & & & 0 & X_{2,k-1} & 0 \\ \vdots & & \iddots & \iddots & 0 & \vdots \\ \vdots & \iddots & \iddots & \ddots & 0 & \vdots \\ 0 & X_{k-1,2} & & & X_{k-1,k-1} & 0 \\ X_{k,1} & 0 & \dots & \dots & 0 & X_{k,k} \end{bmatrix}.$$

For $l = k - j + 1 > j$ the spectrum of every subpencil

$$\lambda \begin{bmatrix} 0 & E_{j,l} & 0 & 0 \\ E_{l,j} & E_{l,l} & 0 & 0 \\ 0 & 0 & 0 & E_{l,j}^* \\ 0 & 0 & E_{j,l}^* & E_{l,l}^* \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & G_{j,l} \\ 0 & 0 & G_{j,l}^* & G_{l,l} \\ 0 & H_{j,l} & 0 & 0 \\ H_{j,l}^* & H_{l,l} & 0 & 0 \end{bmatrix},$$

is equal to $\{\lambda_j, -\lambda_j, \overline{\lambda}_j, -\overline{\lambda}_j\}$ *and for different indices* j *the spectra of the corresponding subpencils are disjoint. Here we allow real and purely imaginary eigenvalues.*

Furthermore, if n *is odd, then* k *must be odd and for* $r = \frac{k+1}{2}$ *the spectrum of the subpencil*

$$\lambda \begin{bmatrix} E_{r,r} & 0 \\ 0 & E_{r,r}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{r,r} \\ H_{r,r} & 0 \end{bmatrix}$$

is $\{\lambda_r, -\lambda_r\}$ *and it is disjoint from the spectra of the other subpencils. In particular,* λ_r *is real or purely imaginary or equal to* ∞.

Before formulating and proving the main result of this section we will give some technical lemmas and introduce some further notation.

Definition 5.3 *Let* $H \in \mathbb{C}^{n\times n}$ *be an Hermitian matrix that has* ν_+ *positive,* ν_- *negative and* ν_0 *zero eigenvalues. We call the triple* $\mathrm{Ind}(H) = (\nu_+, \nu_-, \nu_0)$ *the* inertia index *of* H.

Lemma 5.4 *(Lemma 3 in [15].) Let* $H \in \mathbb{C}^{n\times n}$ *be Hermitian with inertia index* $\mathrm{Ind}(H) = (\nu_+, \nu_-, \nu_0)$. *Then* H *is congruent to a lower anti-triangular matrix if and only if* $|\nu_+ - \nu_-| \leq \nu_0$ *when* n *is even or* $|\nu_+ - \nu_-| \leq \nu_0 + 1$ *when* n *is odd.*

Definition 5.5 *Let* $H \in \mathbb{C}^{n\times n}$ *be Hermitian with* $\mathrm{Ind}(H) = (\nu_+, \nu_-, \nu_0)$. *We say that* H *satisfies the* index condition *if* $|\nu_+ - \nu_-| \leq \nu_0$ *when* n *is even or* $|\nu_+ - \nu_-| \leq \nu_0 + 1$ *when* n *is odd.*

Thus, H satisfies the index condition if and only if it is congruent to an anti-triangular matrix.

Remark 5.6 Let $\mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be Γ-selfadjoint and Hamiltonian as in (12). If $\mathcal{A}$ is in anti-triangular form, then the pencil $\lambda I - \mathcal{A}$ is J, Γ-congruent to a pencil in anti-triangular form via

$$\begin{bmatrix} I_n & 0 \\ 0 & Z_n \end{bmatrix} (\lambda I_{2n} - \mathcal{A}) \begin{bmatrix} Z_n & 0 \\ 0 & I_n \end{bmatrix} = \lambda \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} - \begin{bmatrix} 0 & G \\ ZHZ & 0 \end{bmatrix}.$$

The canonical forms in Theorems 3.2, 4.6 and Remark 5.6 lead to a characterization of all possible subpencils that represent structured Kronecker blocks of the structured pencil $\lambda\mathcal{E} - \mathcal{A}$. With every type of block we will also list the inertia indices. In all cases in the following proposition, $\delta, \varepsilon \in \{1, -1\}$. We use different letters to indicate from which case in Theorems 3.2, 4.6 the structure index comes.

Corollary 5.7 *Let* $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ *be an* ∞*-regular,* Γ*-selfadjoint, and skew-Hamiltonian/Hamiltonian pencil. Then there exists* $\mathcal{W} \in \mathbb{GP}_{2n}$ *such that*

$$(\Gamma^{-1}\mathcal{W}^*\Gamma)(\lambda\mathcal{E} - \mathcal{A})\mathcal{W} = (J^{-1}\mathcal{W}^*J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{W} = \lambda \begin{bmatrix} \tilde{\mathcal{E}} & 0 \\ 0 & \tilde{\mathcal{E}} \end{bmatrix} - \begin{bmatrix} 0 & \tilde{\mathcal{G}} \\ \tilde{\mathcal{H}} & 0 \end{bmatrix},$$

where

$$\tilde{\mathcal{E}} = \mathrm{diag}(E_1, \ldots, E_k), \quad \tilde{\mathcal{G}} = \mathrm{diag}(G_1, \ldots, G_k), \quad \tilde{\mathcal{H}} = \mathrm{diag}(H_1, \ldots, H_k),$$

are all Hermitian, and for every j*, the pencil* $\lambda \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix} - \begin{bmatrix} 0 & G_j \\ H_j & 0 \end{bmatrix}$ *has one and only one of the following forms.*

a. *Even sized blocks associated with the eigenvalue* 0*: The form is either*

$$\lambda\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} := \lambda\begin{bmatrix} Z_p & 0 \\ 0 & Z_p \end{bmatrix} - \begin{bmatrix} 0 & \varepsilon Z_p \\ \varepsilon Z_p \mathcal{J}_p(0) & 0 \end{bmatrix},$$

with inertia indices $\mathrm{Ind}(G) = (q, q, 0)$ *and* $\mathrm{Ind}(\varepsilon H) = (q, q-1, 1)$ *if* $p = 2q$*, and* $\mathrm{Ind}(\varepsilon G) = (q+1, q, 0)$ *and* $\mathrm{Ind}(H) = (q, q, 1)$ *if* $p = 2q+1$*, or*

$$\lambda\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} := \lambda\begin{bmatrix} Z_p & 0 \\ 0 & Z_p \end{bmatrix} - \begin{bmatrix} 0 & \varepsilon Z_p \mathcal{J}_p(0) \\ \varepsilon Z_p & 0 \end{bmatrix},$$

with inertia indices $\mathrm{Ind}(\varepsilon G) = (q, q-1, 1)$ *and* $\mathrm{Ind}(H) = (q, q, 0)$ *if* $p = 2q$*, and* $\mathrm{Ind}(G) = (q, q, 1)$ *and* $\mathrm{Ind}(\varepsilon H) = (q+1, q, 0)$ *if* $p = 2q+1$*.*

b. *Paired odd sized blocks associated with the eigenvalue* 0*:*

$$\lambda\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} := \lambda\begin{bmatrix} Z_{2q+1} & 0 \\ 0 & Z_{2q+1} \end{bmatrix} - \begin{bmatrix} 0 & Z_{2q+1}\mathcal{J}_{2q+1}(0) \\ Z_{2q+1}\mathcal{J}_{2q+1}(0) & 0 \end{bmatrix},$$

with inertia indices $\mathrm{Ind}(G) = \mathrm{Ind}(H) = (q, q, 1)$*.*

c. *Blocks associated with a real eigenvalue pair* $\alpha, -\alpha$*, where* $\alpha > 0$*:*

$$\lambda\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} := \lambda\begin{bmatrix} Z_p & 0 \\ 0 & Z_p \end{bmatrix} - \begin{bmatrix} 0 & \varepsilon Z_p \mathcal{J}_p(\alpha) \\ \varepsilon Z_p \mathcal{J}_p(\alpha) & 0 \end{bmatrix},$$

with inertia indices $\mathrm{Ind}(G) = \mathrm{Ind}(H) = (q, q, 0)$ *if* $p = 2q$ *or* $\mathrm{Ind}(\varepsilon G) = \mathrm{Ind}(\varepsilon H) = (q+1, q, 0)$ *if* $p = 2q+1$*.*

d. *Blocks associated with a purely imaginary eigenvalue pair* $i\alpha, -i\alpha$*, where* $\alpha > 0$*:*

$$\lambda\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} := \lambda\begin{bmatrix} Z_p & 0 \\ 0 & Z_p \end{bmatrix} - \begin{bmatrix} 0 & -\delta Z_p \mathcal{J}_p(\alpha) \\ \delta Z_p \mathcal{J}_p(\alpha) & 0 \end{bmatrix},$$

with inertia indices $\mathrm{Ind}(G) = \mathrm{Ind}(H) = (q, q, 0)$ *if* $p = 2q$*, or* $\mathrm{Ind}(\delta G) = (q, q+1, 0)$ *and* $\mathrm{Ind}(\delta H) = (q+1, q, 0)$ *if* $p = 2q+1$*.*

e. *Blocks associated with a quadruple of finite eigenvalues* $\alpha, \overline{\alpha}, -\alpha, -\overline{\alpha}$*, where* $\alpha^2 \notin \mathbb{R}$*:*

$$\lambda\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} := \lambda\begin{bmatrix} Z_{2q} & 0 \\ 0 & Z_{2q} \end{bmatrix} - \left[\begin{array}{cc|cc} 0 & 0 & 0 & Z_q\mathcal{J}_q(\overline{\alpha}) \\ 0 & 0 & Z_q\mathcal{J}_q(\alpha) & 0 \\ \hline 0 & Z_q\mathcal{J}_q(\overline{\alpha}) & 0 & 0 \\ Z_q\mathcal{J}_q(\alpha) & 0 & 0 & 0 \end{array}\right],$$

with inertia indices $\mathrm{Ind}(G) = \mathrm{Ind}(H) = (q, q, 0)$*.*

f. *Paired even sized blocks associated with the eigenvalue* ∞*:*

$$\lambda\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} := \lambda\begin{bmatrix} Z_{2q}\mathcal{J}_{2q}(0) & 0 \\ 0 & Z_{2q}\mathcal{J}_{2q}(0) \end{bmatrix} - \begin{bmatrix} 0 & Z_{2q} \\ Z_{2q} & 0 \end{bmatrix},$$

with inertia indices $\mathrm{Ind}(G) = \mathrm{Ind}(H) = (q, q, 0)$*.*

g. *Paired odd sized blocks associated with the eigenvalue* ∞*:*

$$\lambda\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} := \lambda\begin{bmatrix} \varepsilon Z_{2q+1}\mathcal{J}_{2q+1}(0) & 0 \\ 0 & \varepsilon Z_{2q+1}\mathcal{J}_{2q+1}(0) \end{bmatrix} - \begin{bmatrix} 0 & \varepsilon Z_{2q+1} \\ \varepsilon Z_{2q+1} & 0 \end{bmatrix},$$

with inertia indices $\mathrm{Ind}(\varepsilon G) = \mathrm{Ind}(\varepsilon H) = (q+1, q, 0)$*.*

Proof. These block forms follow directly from Theorems 4.6, 3.2 and Remark 5.6. The assertion on the inertia indices of the blocks G and H follows easily from Lemma 6 in [15] $\square$

Note that the matrices $Z_j\mathcal{J}(\alpha)_j$ are lower *anti-bidiagonal* and matrices Z_j are lower *anti-diagonal.* So in all cases E, H, G are either lower anti-bidiagonal or lower anti-diagonal.

In order to derive necessary and sufficient conditions so that $\lambda\mathcal{E} - \mathcal{A}$ is J, Γ-congruent to a anti-triangular form, we assemble these subpencils together and we frequently use the following transformation.

Remark 5.8 Let F, M be both lower anti-triangular and partitioned as

$$F = \begin{bmatrix} 0 & 0 & F_{1,3} \\ 0 & f_{2,2} & f_{2,3} \\ F_{3,1} & f_{3,2} & F_{3,3} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & M_{1,3} \\ 0 & m_{2,2} & m_{2,3} \\ M_{3,1} & m_{3,2} & M_{3,3} \end{bmatrix},$$

where $F_{1,3}$ and $F_{3,1}$ ($M_{1,3}$ and $M_{3,1}$) are square and have the same size, respectively and furthermore $f_{2,2}$ ($m_{2,2}$) is either a scalar if the size of F (M) is odd or is void if the size is even. Then there exists a permutation matrix P such that

$$H := \begin{bmatrix} F & 0 \\ 0 & M \end{bmatrix} = P^* \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & F_{1,3} \\ 0 & 0 & 0 & 0 & M_{1,3} & 0 \\ \hline 0 & 0 & f_{2,2} & 0 & 0 & f_{2,3} \\ 0 & 0 & 0 & m_{2,2} & m_{2,3} & 0 \\ \hline 0 & M_{3,1} & 0 & m_{3,2} & M_{3,3} & 0 \\ F_{3,1} & 0 & f_{3,2} & 0 & 0 & F_{3,3} \end{array}\right] P. \tag{17}$$

Obviously, if H is Hermitian then this block lower anti-triangular form is still Hermitian.

Then H is congruent to a lower anti-triangular form if $f_{2,2}$, $m_{2,2}$ are as in the following cases.

Case 1: If $f_{2,2} = 0$ this is obvious and if $m_{2,2} = 0$, then by a block permutation we switch the roles of F and M in H. In this case F or M, respectively, has odd size.

Case 2: If $f_{2,2}$ or $m_{2,2}$ is void, then $f_{2,3}$, $f_{3,2}$ or $m_{2,3}$, $m_{3,2}$, respectively, are void and this is also obvious. In this case F or M, respectively, has even size.

Case 3: If $f_{2,2}m_{2,2} < 0$, then let $X = \begin{bmatrix} 1 & 0 \\ \sqrt{-\frac{f_{2,2}}{m_{2,2}}} & 1 \end{bmatrix}$. Then it is easy to see that

$$X^* \begin{bmatrix} f_{2,2} & 0 \\ 0 & m_{2,2} \end{bmatrix} X = \begin{bmatrix} 0 & m_{2,2}\sqrt{-\frac{f_{2,2}}{m_{2,2}}} \\ m_{2,2}\sqrt{-\frac{f_{2,2}}{m_{2,2}}} & m_{2,2} \end{bmatrix}. \tag{18}$$

Applying this transformation to the matrix in (17), we can reduce H to a anti-triangular form.

It should be noted, that when $F_{1,3}, F_{3,1}, M_{1,3}, M_{3,1}$ are all nonsingular, then these three cases give necessary conditions for H to be congruent to a lower anti-triangular matrix.

Another useful permutation is

$$H := \begin{bmatrix} F & 0 \\ 0 & M \end{bmatrix} = P^* \begin{bmatrix} 0 & 0 & 0 & F_{1,3} \\ 0 & 0 & 0 & f_{2,3} \\ 0 & 0 & M & 0 \\ F_{3,1} & f_{3,2} & 0 & F_{3,3} \end{bmatrix} P, \tag{19}$$

if $f_{2,2} = 0$ and

$$H := \begin{bmatrix} F & 0 \\ 0 & M \end{bmatrix} = P^* \begin{bmatrix} 0 & 0 & F_{1,3} \\ 0 & M & 0 \\ F_{3,1} & 0 & F_{3,3} \end{bmatrix} P, \tag{20}$$

if f_{22} is void. When M is already lower anti-triangular, in both cases H is congruent to a lower anti-triangular form.

Remark 5.9 In order to compute a lower anti-triangular form we perform J, Γ-congruent transformations to the pencil

$$\lambda \begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} - \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}$$

with block diagonal matrices $\operatorname{diag}(U, V)$. This is equivalent to performing transformations

$$U^*EV, \quad U^*GU, \quad V^*HV$$

on the matrix triple E, G, H. We will often use the following special transformations.

1. If

$$E = \operatorname{diag}(E_1, E_2), \quad G = \operatorname{diag}(G_1, G_2), \quad H = \operatorname{diag}(H_1, H_2),$$

then by taking $U = I$ and $V = \operatorname{diag}(I, -I)$, we can transform the matrix triple to

$$E = \operatorname{diag}(E_1, -E_2), \quad G = \operatorname{diag}(G_1, G_2), \quad H = \operatorname{diag}(H_1, H_2).$$

This means that we can freely change the sign of E_2, and analogously, we can also freely change the sign of E_1.

2. If

$$E = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix}, \quad H = \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix},$$

where $\gamma_1, \gamma_2, g, h \in \mathbb{C}$, then taking $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we obtain that

$$EX = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix}, \quad X^*HX = \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix}$$

are all in lower anti-triangular form.

3. If

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & h \\ 0 & 0 & 0 \\ \overline{h} & 0 & 0 \end{bmatrix},$$

where $\varepsilon = 1$ or $\varepsilon = -1$ and $h \in \mathbb{C}$, then taking $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ it follows that

$$X^*EY = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad X^*GX = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varepsilon \\ 0 & -\varepsilon & -\varepsilon \end{bmatrix}, \quad Y^*HY = \begin{bmatrix} 0 & 0 & h \\ 0 & 0 & 0 \\ \overline{h} & 0 & 0 \end{bmatrix} \tag{21}$$

are all in anti-triangular form. Moreover, the middle anti-diagonal (also diagonal) element of the transformed matrices X^*GX and Y^*HY is 0.

By Proposition 5.2, Theorem 4.6, and Remark 5.8, it follows that $\lambda\mathcal{E}-\mathcal{A}$ is J,Γ-congruent to a lower anti-triangular form if and only if every subpencil from the structured canonical form that combines the whole multiplicity of a quadruple $\{\alpha,-\alpha,\overline{\alpha},-\overline{\alpha}\}$ of non-real or non-purely imaginary eigenvalues or a pair of eigenvalues $\{\alpha,-\alpha\}$ with $\alpha^2\in\mathbb{R}\cup\{\infty\}$ is J,Γ-congruent to a lower anti-triangular form. Based on this fact we can use the subpencils in Corollary 5.7 to find the conditions for the existence of a lower anti-triangular form.

Lemma 5.10 *Let* $\lambda\mathcal{E}-\mathcal{A}\in\mathbb{C}^{2n\times 2n}$ *be an* ∞*-regular,* Γ*-selfadjoint, and skew-Hamiltonian/Hamiltonian pencil that has only a single eigenvalue quadruple* $\{\alpha,-\alpha,\overline{\alpha},-\overline{\alpha}\}$ *with* $\operatorname{Re}\alpha\operatorname{Im}\alpha\neq 0$*, or a single pair of eigenvalues* $\{\alpha,-\alpha\}$*, or* $\{i\alpha,-i\alpha\}$ *with* $\alpha>0$ *or a single eigenvalue* 0 *or* ∞*. Suppose that the pencil is* J,Γ*-congruent to the structured canonical form*

$$\lambda\tilde{\mathcal{E}}-\tilde{\mathcal{A}}=\lambda\begin{bmatrix} E & 0\\ 0 & E\end{bmatrix}-\begin{bmatrix} 0 & G\\ H & 0\end{bmatrix},$$

with

$$E=\operatorname{diag}(E_1,\ldots,E_k),\quad G=\operatorname{diag}(G_1,\ldots,G_k),\quad H=\operatorname{diag}(H_1,\ldots,H_k)$$

and every

$$\lambda\begin{bmatrix} E_j & 0\\ 0 & E_j\end{bmatrix}-\begin{bmatrix} 0 & G_j\\ H_j & 0\end{bmatrix},$$

has one of the forms as in Corollary 5.7. Then $\lambda\mathcal{E}-\mathcal{A}$ *is* J,Γ*-congruent to a lower triangular form if and only if both* G *and* H *satisfy the index condition.*

Proof. The necessity is clear, so we only prove the sufficiency.

We consider five different cases, based on the types of eigenvalues.

1. For blocks as in Corollary 5.7 e., every E_j,G_j,H_j is are already lower anti-triangular and has even size. Applying the permutation (20) simultaneously to the triple several times, we obtain the lower anti-triangular forms for E,G,H. Obviously G and H satisfy the index condition.

2. For an eigenvalue pair $\{\alpha,-\alpha\}$ and $\alpha>0$, by Corollary 5.7 c. it follows that

$$E_j=Z_{p_j},\quad G_j=H_j=\varepsilon_j Z_{p_j}\mathcal{J}_{p_j}(\alpha).$$

Let $p_j=2q_j+1$ for $j=1,\ldots,l$ and $p_j=2q_j$ for $j=l+1,\ldots,k$. For $\lambda\mathcal{E}-\mathcal{A}$ in lower anti-triangular form it is necessary that G, H must satisfy the index condition. Since G and H are nonsingular this means that

$$\nu_+(G)-\nu_-(G)=\nu_+(H)-\nu_-(H)=0$$

if n is even and

$$|\nu_+(G)-\nu_-(G)|=|\nu_+(H)-\nu_-(H)|=1$$

if n is odd. On the other hand by Corollary 5.7 c.

$$\nu_+(G)-\nu_-(G)=\nu_+(H)-\nu_-(H)=\sum_{j=1}^{l}\varepsilon_j.$$

Hence if n is even, then $\sum_{j=1}^{l}\varepsilon_j=0$, which implies that l, the number of the odd sized Jordan blocks must be even and the numbers of the structure indices with $\varepsilon_j=1$ and $\varepsilon_j=-1$ must be equal. If n is odd then l is odd and all but one of the ε_j must occur in $1,-1$ pairs.

To show that this is also sufficient, we consider the cases that n is odd or even separately.

If n is even, then l is even and we can permute the blocks in the original pencil $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ such that the canonical blocks of odd size are paired into $l/2$ subpencils as in

$$\lambda \begin{bmatrix} E_i & 0 & 0 & 0 \\ 0 & E_j & 0 & 0 \\ 0 & 0 & E_i & 0 \\ 0 & 0 & 0 & E_j \end{bmatrix} - \begin{bmatrix} 0 & 0 & G_i & 0 \\ 0 & 0 & 0 & G_j \\ H_i & 0 & 0 & 0 \\ 0 & H_j & 0 & 0 \end{bmatrix},$$

where $\varepsilon_i = 1$, $\varepsilon_j = -1$. By Remark 5.9 we now consider a transformation on this matrix triple. Applying (17) and (18), the triple can be transformed to lower anti-triangular form.

In this way we can get $l/2$ even sized matrix triples which are all lower anti-triangular. Joining these and the matrix triples associated with even sized canonical blocks, using (20) again we get the lower anti-triangular form.

The case n is odd is similar to the even case. The only difference is that after pairing there is still one odd sized matrix triple left. But applying (20) to assemble the whole lower anti-triangular form, the only difference is that the odd sized blocks should be put in the bottom as block M in (20).

3. The proof for pairs of purely imaginary eigenvalues is the same as that for **2**.

4. For zero eigenvalues by Corollary 5.7 a. and b., the matrix triple E_j, G_j, H_j has three possible forms

i) $E_j = Z_{p_j}$, $\quad G_j = \varepsilon_j Z_{p_j}$, $\quad H_j = \varepsilon_j Z_{p_j} \mathcal{J}_{p_j}(0)$;

ii) $E_j = Z_{p_j}$, $\quad G_j = \varepsilon_j Z_{p_j} \mathcal{J}_{p_j}(0)$, $\quad H_j = \varepsilon_j Z_{p_j}$;

iii) $E_j = Z_{2q_j+1}$, $\quad G_j = H_j = Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0)$.

Forms i) and ii) are associated with even sized canonical blocks and form iii) is associated with odd sized canonical blocks. Assume that $\lambda\mathcal{E} - \mathcal{A}$ has k_1 and k_2 canonical blocks of even size with respect to form i) and ii), respectively and k_3 blocks of odd size and form iii). Without loss of generality assume that the matrix triples E_j, G_j, H_j have form i), ii), and iii) for $j = 1, \ldots, k_1$, $j = k_1 + 1, \ldots, k_1 + k_2$, and $j = k_1 + k_2, \ldots, k (= k_1 + k_2 + k_3)$, respectively. Moreover, assume that $p_j = 2q_j$ for $j = 1, \ldots, k_{11}$ and $j = k_1 + 1, \ldots, k_1 + k_{21}$ and $p_j = 2q_j + 1$ for $j = k_{11} + 1, \ldots, k_1$ and $j = k_1 + k_{21} + 1, \ldots, k_1 + k_2$, i.e., there are k_{11} matrix triples of form i) with even size and $k_1 - k_{11}$ of this form with odd size, and there are k_{21} matrix triples of form ii) with even size and $k_2 - k_{21}$ of this form with odd size.

From these block forms we get the following relations for the inertia indices of G and H.

$$\begin{aligned} \nu_+(G) - \nu_-(G) &= \sum_{j=1}^{k} (\nu_+(G_j) - \nu_-(G_j)) = \sum_{j=1}^{k_1 - k_{11}} \varepsilon_{k_{11}+j} + \sum_{j=1}^{k_{21}} \varepsilon_{k_1+j}, \\ \nu_0(G) &= \sum_{j=1}^{k} \nu_0(G_j) = k_2 + k_3, \\ \nu_+(H) - \nu_-(H) &= \sum_{j=1}^{k} (\nu_+(H_j) - \nu_-(H_j)) = \sum_{j=1}^{k_{11}} \varepsilon_j + \sum_{j=1}^{k_2 - k_{21}} \varepsilon_{k_1+k_{21}+j}, \\ \nu_0(H) &= \sum_{j=1}^{k} \nu_0(G_j) = k_1 + k_3. \end{aligned}$$

If H and G satisfy the index condition and if n is even, then

$$\left|\sum_{j=1}^{k_1-k_{11}} \varepsilon_{k_{11}+j} + \sum_{j=1}^{k_{21}} \varepsilon_{k_1+j}\right| \leq k_2 + k_3; \quad \left|\sum_{j=1}^{k_{11}} \varepsilon_j + \sum_{j=1}^{k_2-k_{21}} \varepsilon_{k_1+k_{21}+j}\right| \leq k_1 + k_3; \tag{22}$$

and if n is odd, then

$$\left|\sum_{j=1}^{k_1-k_{11}} \varepsilon_{k_{11}+j} + \sum_{j=1}^{k_{21}} \varepsilon_{k_1+j}\right| \leq k_2 + k_3 + 1; \quad \left|\sum_{j=1}^{k_{11}} \varepsilon_j + \sum_{j=1}^{k_2-k_{21}} \varepsilon_{k_1+k_{21}+j}\right| \leq k_1 + k_3 + 1.$$

We now show that these conditions are sufficient to construct the lower anti-triangular form for $\lambda\mathcal{E} - \mathcal{A}$. We just consider the case that n is even. If n is odd, then we can use the construction used in **2**.

Our main task is to find the pairing technique to transform the odd sized matrix triples into even sized lower anti-triangular matrix triples. Once this is done we can assemble these triples and the remaining even sized triples for even sized canonical forms to get the final lower anti-triangular form.

The odd sized matrix triples are distributed as follows. $k_1 - k_{11}$ triples of form i), $k_2 - k_{21}$ triples of form ii) and k_3 triples of form iii). For the odd sized matrix triples of form i) the difference between the number of index $\varepsilon_j = 1$ and -1 is $l_1 = \left|\sum_{j=1}^{k_1-k_{11}} \varepsilon_{k_{11}+j}\right|$. For the odd sized matrix triples of form ii) the difference is $l_2 = \left|\sum_{j=1}^{k_2-k_{21}} \varepsilon_{k_1+k_{21}+j}\right|$. Without loss of generality we assume that $l_1 \geq l_2$. We now use the following steps to pair and transform the odd sized matrix triples.

(α) Let E_i, H_i, G_i and E_j, H_j, G_j be of form i) and the corresponding structure indices satisfy $\varepsilon_i = -\varepsilon_j$ (if there is any such pair). Recall that by Remark 5.9 we can freely change the signs of the diagonal blocks of the block diagonal matrix $\tilde{\mathcal{E}}$. Thus, we may consider a triple of the form

$$\begin{bmatrix} Z_{2q_i+1} & 0 \\ 0 & -Z_{2q_j+1} \end{bmatrix}, \begin{bmatrix} \varepsilon_i Z_{2q_i+1} & 0 \\ 0 & \varepsilon_j Z_{2q_j+1} \end{bmatrix}, \begin{bmatrix} \varepsilon_i Z_{2q_i+1}\mathcal{J}_{2q_i+1}(0) & 0 \\ 0 & \varepsilon_j Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0) \end{bmatrix}.$$

By using 17) and (18) it is obviouly possible to transform this triple to a triple of even sized blocks in anti-triangular form. Having used this technique for all possible such pairs we now still have l_1 odd sized matrix triples of form i).

(β) If $l_1 = 0$, then by assumption $l_2 = 0$. Then the odd sized matrix triple of form ii) can be also paired such that the signs of the structure indices is opposite. We can use the same method as in step (α) to transform all such pairs to even sized matrix triples in lower anti-triangular form. Now the only odd sized triples are of form iii). Since n is even the number of such triples must be even. So we can pair them and for each pair we can apply (17) and the transformation in case 2 of Remark 5.9 to the triple

$$\begin{bmatrix} Z_{2q_i+1} & 0 \\ 0 & Z_{2q_j+1} \end{bmatrix}, \begin{bmatrix} Z_{2q_i+1}\mathcal{J}_{2q_i+1}(0) & 0 \\ 0 & Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0) \end{bmatrix},$$

$$\begin{bmatrix} Z_{2q_i+1}\mathcal{J}_{2q_i+1}(0) & 0 \\ 0 & Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0) \end{bmatrix}.$$

to obtain an even sized lower anti-triangular matrix triple as

$$\left[\begin{array}{c|cc|c} 0 & 0 & 0 & Z_{q_i+q_i} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline Z_{q_i+q_j} & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{c|cc|c} 0 & 0 & 0 & F \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ \hline F^* & * & * & * \end{array}\right], \quad \left[\begin{array}{c|cc|c} 0 & 0 & 0 & M \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ \hline M^* & * & * & * \end{array}\right],$$

where F, M are lower anti-triangular. Finally, we apply (20) to all these even sized matrix triples to get the lower anti-triangular form for $\lambda\mathcal{E} - \mathcal{A}$.

(γ) If $l_1 \geq l_2 > 0$, we can pair an odd sized matrix triple of form i) and an odd sized matrix triple of form ii). In this way we form l_2 pairs. For each pair with E_j, G_j, H_j of form i) and E_i, G_i, H_i of form ii) we consider a simultaneous permutation on

$$\left[\begin{array}{cc} Z_{2q_i+1} & 0 \\ 0 & Z_{2q_j+1} \end{array}\right], \quad \left[\begin{array}{cc} \varepsilon_i Z_{2q_i+1}\mathcal{J}_{2q_i+1}(0) & 0 \\ 0 & \varepsilon_j Z_{2q_j+1} \end{array}\right], \quad \left[\begin{array}{cc} \varepsilon_i Z_{2q_i+1} & 0 \\ 0 & \varepsilon_j Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0) \end{array}\right].$$

Using (17) and the transformation in case 2 of Remark 5.9 again, we get a matrix triple

$$\left[\begin{array}{c|cc|c} 0 & 0 & 0 & Z_{q_i+q_i} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline Z_{q_i+q_j} & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{c|cc|c} 0 & 0 & 0 & F \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & \varepsilon_j & * \\ \hline F^* & * & * & * \end{array}\right], \quad \left[\begin{array}{c|cc|c} 0 & 0 & 0 & M \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & \varepsilon_i & * \\ \hline M^* & * & * & * \end{array}\right],$$

where F, M are lower anti-triangular. Now we still have $l_1 - l_2$ odd sized matrix triples of form i), $k_2 - k_{21} - l_2$ triples of form ii), and k_3 triples of form iii).

(δ) If $l_1 = l_2$ then we can pair the remaining $k_2 - k_{21} - l_2$ odd sized matrix triples of form ii) with structure indices in ± 1 pattern. Also, k_3 is even and we can pair the triples of form iii). Using the method in step (β) we can get the lower anti-triangular form.

(ϵ) If $l_1 > l_2$ we pair a remaining odd sized matrix triple of form i) with a matrix triple of form iii) (if there is any). Let E_j, G_j, H_j be a remaining triple of form i) and E_i, G_i, H_i of form iii). As in step (γ) the paired triple can be transformed to an even sized matrix triple in lower anti-triangular form:

$$\left[\begin{array}{c|cc|c} 0 & 0 & 0 & Z_{q_i+q_i} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline Z_{q_i+q_j} & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{c|cc|c} 0 & 0 & 0 & F \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & \varepsilon_j & * \\ \hline F^* & * & * & * \end{array}\right], \quad \left[\begin{array}{c|cc|c} 0 & 0 & 0 & M \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ \hline M^* & * & * & * \end{array}\right].$$

We can get a total number of $\min\{k_3, l_1 - l_2\}$ of such triples.

(ε) If $l_1 - l_2 \leq k_3$, we still have $k_2 - k_{21} - l_2$ odd sized matrix triples of form ii) which can be paired and remaining $k_3 - (l_1 - l_2)$ matrix triples of form iii). Since n is even, based on the block sizes it is obvious that $k_3 - (l_1 - l_2)$ is even. So again we can apply step (β) to get the lower anti-triangular form.

(ζ) If $l_1 - l_2 > k_3$ then there are still $l_1 - l_2 - k_3$ odd sized matrix triples of form i) and $k_2 - k_{21} - l_2$ (which is even) odd sized matrix triples of form ii). Similarly $l_1 - l_2 - k_3$ must be even. We now use two of such triples and one even sized matrix triple of form ii) with opposite structure index to construct an even sized anti-triangular form. First, let E_j, G_j, H_j be a remaining triple of form i) and E_i, G_i, H_i of form iii) with $\varepsilon_i = -\varepsilon_j$. We consider permutations on

$$\left[\begin{array}{cc} Z_{2q_i} & 0 \\ 0 & Z_{2q_j+1} \end{array}\right], \quad \left[\begin{array}{cc} \varepsilon_i Z_{2q_i}\mathcal{J}_{2q_i}(0) & 0 \\ 0 & \varepsilon_j Z_{2q_j+1} \end{array}\right], \quad \left[\begin{array}{cc} \varepsilon_i Z_{2q_i} & 0 \\ 0 & \varepsilon_j Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0) \end{array}\right].$$

Using (20) we get

$$\begin{bmatrix} 0 & 0 & Z_{q_i} \\ 0 & Z_{2q_j+1} & 0 \\ Z_{q_i} & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \varepsilon_i Z_{q_i}\mathcal{J}_{q_i}(0) \\ 0 & \varepsilon_j Z_{2q_j+1} & 0 \\ \varepsilon_i Z_{q_i}\mathcal{J}_{q_i}(0) & 0 & \varepsilon_i e_1^* e_1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 & \varepsilon_i Z_{q_i} \\ 0 & \varepsilon_j Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0) & 0 \\ \varepsilon_i Z_{q_i} & 0 & 0 \end{bmatrix},$$

where e_1 is the first unit vector. Partitioning Z_{2q_j+1}, $\varepsilon_j Z_{2q_j+1}$, and $\varepsilon_j Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0)$ into 3×3 block forms with middle anti-diagonal block 1×1, the matrices in the above triple turn out in 5×5 block forms. Permuting the last 2 block rows and columns and the first 2 block rows and columns simultaneously, then with the structures of Z and $\mathcal{J}(0)$ we get a new triple of the form

$$\left[\begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & Z \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & F \\ \hline 0 & 0 & 0 & 0 & \varepsilon_i e_1^* \\ 0 & 0 & \varepsilon_j & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_i & 0 \\ \hline F^* & \varepsilon_i e_1 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & M \\ \hline 0 & 0 & 0 & \varepsilon_i & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_j e_1^* \\ 0 & \varepsilon_i & 0 & 0 & 0 \\ \hline M^* & 0 & \varepsilon_j e_1 & 0 & 0 \end{array}\right],$$

where F, M are lower anti-triangular. Since $\varepsilon_i\varepsilon_j = -1$, applying transformation (21) to the middle blocks, we have an odd sized matrix triple $\tilde{E}, \tilde{G}, \tilde{H}$ in lower anti-triangular form. Moreover the entries on the middle of the anti-diagonals of $\tilde{G}, \tilde{H}$ are zero. Using this fact we can pair another remaining odd sized matrix triple of form i) with $\tilde{E}, \tilde{G}, \tilde{H}$, and apply (17) and case 2 of Remark 5.9 as in (γ) to get an even sized matrix triple in anti-triangular form. Note that all remaining odd sized matrix triples of form i) must have the same structure index. Let m be the number of even sized matrix triples of form ii) with structure index opposite to that of the remaining triples of form i). Then the procedure above can be applied $\min\{m, (l_1 - l_2 - k_3)/2\}$ times.

(η) If $l_1 - l_2 - k_3 \le 2m$, then we only have $k_2 - k_{21} - l_1$ odd sized matrix triples of form ii) left. These can be paired as in (α).

(θ) If $s := l_1 - l_2 - k_3 - 2m > 0$, we still have s (which is even) odd sized matrix triples of form i) and $k_2 - k_{21} - l_2$ (which is also even) odd sized matrix triples of form ii) such that half of their structure indices are 1 and half of them are -1. Without loss of generality we assume $l_1 = \sum_{j=1}^{k_1-k_{11}} \varepsilon_{k_{11}+j}$, i.e., the structure indices of all remaining odd sized matrix triples of form i) are 1. Then m is the number of even sized matrix triples of form ii) with structure indices -1. The index condition (22) now implies that

$$\left|(k_{21} - m) - m + \sum_{j=1}^{k_1-k_{11}} \varepsilon_{k_{11}+j}\right| = l_1 + k_{21} - 2m \le k_2 + k_3$$

or $s \le k_2 - k_{21} - l_2$. We can choose $k_2 - k_{21} - l_2 - s$ (which is even) odd sized matrix triples of form ii) paired with index pattern ± 1. Applying the method in step (α) to each pair, we can get an even sized lower anti-triangular matrix triple. We are then left with s odd sized triples of form ii). Each of the remaining s odd sized matrix triples of form i) can now be paired with one of the remaining s odd sized matrix triples of form ii). Applying the method in step

(γ) we can also get an even sized lower anti-triangular matrix triple. Finally, we only have even sized matrix triples all of them in lower anti-triangular form. Applying (20) to these even sized matrix triples we can get the lower anti-triangular form for $\lambda\mathcal{E} - \mathcal{A}$. the method

5. For the eigenvalue ∞, by Corollary 5.7, f) and g), the matrix triple E_j, G_j, H_j has one of the two forms

$$E_j = Z_{2q_j}\mathcal{J}_{2q_j}(0), \quad G_j = H_j = Z_{2q_j},$$

or

$$E_j = \varepsilon_j Z_{2q_j+1}\mathcal{J}_{2q_j+1}(0), \quad G_j = H_j = \varepsilon_j Z_{2q_j+1}.$$

For n even, if G, H satisfy the index condition, we immediately have that the number of indices 1 and -1 are the same. Hence, we can pair the odd sized matrix triples in ± 1 pattern and apply (17) and (18) simultaneously to the matrices of each triple to transform it to an even sized matrix triple in anti-triangular form. Applying (20) to these triples and the even sized matrix triples for even sized canonical forms we get the lower anti-triangular form of $\lambda\mathcal{E} - \mathcal{A}$. For n odd, the anti-triangular form is constructed analogously. □

We now have all the ingrediences to prove the main result of this section.

Theorem 5.11 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be an ∞-regular, Γ-selfadjoint, and skew-Hamiltonian/Hamiltonian pencil as in (13). Then the following are equivalent:*

1. *There exists a matrix $P \in \mathbb{GP}_{2n}$, such that $J^{-1}P^*J(\lambda\mathcal{E} - \mathcal{A})P$ is in anti-triangular form.*

2. *There exists a unitary matrix $Q \in \mathbb{GP}_{2n}$, such that $J^{-1}Q^*J(\lambda\mathcal{E} - \mathcal{A})Q$ is in anti-triangular form.*

3. *If n is even, then the dimension of the deflating subspace associated with any set $\{\lambda_0, -\lambda_0 | \ \lambda_0^2 \in \mathbb{R} \cup \{\infty\}\}$ of eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$ is a multiple of 4.*

   If n is odd, then the dimension of the deflating subspace associated with any, but exactly one set $\{\lambda_0, -\lambda_0 | \ \lambda_0^2 \in \mathbb{R} \cup \{\infty\}\}$ of eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$ is a multiple of 4.

   Moreover, in both cases for any λ_0 with $\lambda_0^2 \in \mathbb{R} \cup \{\infty\}$, if r is the dimension of the deflating subspace associated with $\{\lambda_0, -\lambda_0\}$ and if the columns of

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}, \quad V_1, V_2 \in \mathbb{C}^{\frac{n}{2}\times\frac{r}{2}} \tag{23}$$

   form a basis of this deflating subspace, then $V_2^*GV_2$ and $V_1^*HV_1$ satisfy the index condition.

Proof. We only consider the case that n is even. The case that n is odd can be shown in an analogous way.

(1. $\Leftrightarrow$ 2.): Let $P = \mathrm{diag}(P_1, P_2) \in \mathbb{GP}_{2n}$, such that $J^{-1}P^*J(\lambda\mathcal{E} - \mathcal{A})P$ is in anti-triangular form and let $P_1 = Q_1R_1$ and $P_2 = Q_2R_2$ be QR-decompositions of P_1 and P_2. Setting $Q = \mathrm{diag}(Q_1, Q_2)$, it is easy to see that

$$J^{-1}Q^*J(\lambda\mathcal{E} - \mathcal{A})Q = J^{-1}\begin{bmatrix} R_1^{-*} & 0 \\ 0 & R_2^{-*} \end{bmatrix} P^*J(\lambda\mathcal{E} - \mathcal{A})P \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix}$$

is still in anti-triangular form. The converse is obvious.

By Proposition 5.2, we may assume that the spectrum of the pencil is $\{\alpha, \overline{\alpha}, -\alpha, -\overline{\alpha}\}$ for some $\alpha \in \mathbb{C} \cup \{\infty\}$. (1. $\Leftrightarrow$ 3.) then follows from Lemma 5.10. $\square$

In this section we have derived necessary and sufficient conditions for the existence of transformations to anti-triangular form. It should be noted, that if the transformation exists, then it can be done with unitary transformations and this is good news, since it opens the perspective for numerically stable algorithms.

On the other hand, we have seen that difficulties may arise from blocks associated with real, purely imaginary, or infinite eigenvalues. But if no reduction to anti-triangular condensed form exists, then either we may weaken the restriction to anti-triangular form or we may allow non-unitary transformations. We study these possibilities in the next section.

6 Reduction to almost anti-triangular form

As shown in Section 5, a reduction to structured Schur form is not always possible for the matrices of the form (12) and the pencils of the form (13). Therefore, one has to allow also non-unitary transformations in a reduction to a condensed form if one wants to preserve both structures. In [1] such a reduction method was introduced for the case of matrices from linear response theory. This method results in a form that displays the eigenvalues and that is obtained by using unitary transformations as well as hyperbolic rotations. In this section, we will generalize this method to the pencil case. Let us start with some technical lemmas that can be easily verified.

Lemma 6.1 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be a regular pencil that is Γ-selfadjoint and skew-Hamiltonian/Hamiltonian. If λ_0 is an eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$ and if $U = [U_1^T,\ U_2^T]^T \neq 0$, with $U_1, U_2 \in \mathbb{C}^{n\times r}$, forms a basis of the right deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ associated with the eigenvalue λ_0, then*

1. *$[U_1^T,\ -U_2^T]^T = \Sigma U$ is a basis of the right deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ associated with the eigenvalue $-\lambda_0$.*
2. *$[U_2^*,\ U_1^*] = (\Gamma U)^*$ is a basis of the left deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ associated with the eigenvalue $\overline{\lambda}_0$.*
3. *$[U_2^*,\ -U_1^*] = (JU)^*$ is a basis of the left deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ associated with the eigenvalue $-\overline{\lambda}_0$.*

Lemma 6.2 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be a regular, Γ-selfadjoint, and skew-Hamiltonian/Hamiltonian pencil determined by matrix triple E, G, H, and let $U = [U_1^T,\ U_2^T]^T$, with $U_1, U_2 \in \mathbb{C}^{n\times r}$, be a basis of the right deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ associated with the eigenvalue λ_0 such that there exist matrices $A, B \in \mathbb{C}^{r\times r}$, with*

$$\mathcal{E}UA = \mathcal{A}UB.$$

Then one of the following cases hold:

1. *If $\lambda_0 = \infty$, then U satisfies*

$$\begin{aligned}
&\mathcal{E}U = \mathcal{A}UB, \quad \mathcal{E}(\Sigma U) = \mathcal{A}(\Sigma U)(-B),\\
&\det U^*(\Gamma\mathcal{A})U \neq 0, \quad \det U^*(J\mathcal{A})U \neq 0,\\
&U^*(\Gamma\mathcal{E})U = B^*U^*(\Gamma\mathcal{A})U = U^*(\Gamma\mathcal{A})UB,\\
&U^*(J\mathcal{E})U = (-B^*)U^*(J\mathcal{A})U = U^*(J\mathcal{A})UB.
\end{aligned}$$

2. If $\lambda_0 = 0$, then U satisfies

$$\begin{aligned}
&\mathcal{E}UA = \mathcal{A}U, \quad \mathcal{E}(\Sigma U)(-A) = \mathcal{A}(\Sigma U),\\
&\det U^*(\Gamma\mathcal{E})U \neq 0, \quad \det U^*(J\mathcal{E})U \neq 0,\\
&U^*(\Gamma\mathcal{A})U = A^*U^*(\Gamma\mathcal{E})U = U^*(\Gamma\mathcal{E})UA,\\
&U^*(J\mathcal{A})U = (-A^*)U^*(J\mathcal{E})U = U^*(J\mathcal{E})UA.
\end{aligned}$$

3. If λ_0 is non-zero real, then U satisfies

$$\begin{aligned}
&\mathcal{E}UA = \mathcal{A}U, \quad \mathcal{E}(\Sigma U)(-A) = \mathcal{A}(\Sigma U),\\
&\det U^*(\Gamma\mathcal{E})U \neq 0, \quad U^*(J\mathcal{E})U = U^*(J\mathcal{A})U = 0,\\
&U^*(\Gamma\mathcal{A})U = A^*U^*(\Gamma\mathcal{E})U = U^*(\Gamma\mathcal{E})UA.
\end{aligned}$$

4. If λ_0 is non-zero purely imaginary, then U satisfies

$$\begin{aligned}
&\mathcal{E}UA = \mathcal{A}U, \quad \mathcal{E}(\Sigma U)(-A) = \mathcal{A}(\Sigma U),\\
&\det U^*(J\mathcal{E})U \neq 0, \quad U^*(\Gamma\mathcal{E})U = U^*(\Gamma\mathcal{A})U = 0,\\
&U^*(J\mathcal{A})U = (-A^*)U^*(J\mathcal{E})U = U^*(J\mathcal{E})UA.
\end{aligned}$$

5. If λ_0 is non-real and non-purely imaginary, then U satisfies

$$\begin{aligned}
&\mathcal{E}UA = \mathcal{A}U, \quad \mathcal{E}(\Sigma U)(-A) = \mathcal{A}(\Sigma U),\\
&U^*(\Gamma\mathcal{E})U = U^*(\Gamma\mathcal{A})U = 0, \quad U^*(J\mathcal{E})U = U^*(J\mathcal{A})U = 0.
\end{aligned}$$

Proof. For any regular pencil $\lambda\mathcal{E} - \mathcal{A}$, if U, W are bases of the right and left deflating subspaces associated with a single eigenvalue λ_0 then

$$\mathcal{E}UA = \mathcal{A}U, \qquad \Lambda(A) = \{\lambda_0\}, \qquad \det W\mathcal{E}U \neq 0,$$

if λ_0 is finite, and

$$\mathcal{E}U = \mathcal{A}UB, \qquad \Lambda(B) = \{0\}, \qquad \det W\mathcal{A}U \neq 0,$$

if $\lambda_0 = \infty$. Here, $\Lambda(M)$ denotes the spectrum of the matrix M.

Furthermore, if U, W are bases of the right and left deflating subspaces of $\lambda\mathcal{E} - \mathcal{A}$ associated with two different finite eigenvalues λ_0, μ_0, respectively, then $W\mathcal{E}U = W\mathcal{A}U = 0$, see [6].

With these facts and Lemma 6.1, the relations in Lemma 6.2 are easy to verify. □

Lemma 6.3 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be a regular, Γ-selfadjoint, and skew-Hamiltonian/Hamiltonian pencil. Furthermore, let X form a basis of the eigenspace associated with the eigenvalue $\lambda_0 \in \mathbb{R} \cup (i\mathbb{R}) \cup \{\infty\}$, i.e., $\lambda_0\mathcal{E}X = \mathcal{A}X$ if λ_0 is finite or $\mathcal{E}X = 0$ and $\mathcal{A}X$ is of full column rank if $\lambda_0 = \infty$. Then λ_0 is semi-simple, i.e., the sizes of Kronecker blocks are all* 1×1, *if and only if the following conditions hold:*

1. If $\lambda_0 \neq 0$ is real, then $X^*(\Gamma\mathcal{A})X = \lambda_0 X^*(\Gamma\mathcal{E})X$ is nonsingular.

2. *If $\lambda_0 \neq 0$ is purely imaginary, then $X^*(J\mathcal{A})X = \lambda_0 X^*(J\mathcal{E})X$ is nonsingular.*

3. *If $\lambda_0 = 0$, then $X^*(\Gamma\mathcal{E})X$ and $X^*(J\mathcal{E})X$ are nonsingular.*

4. *If $\lambda_0 = \infty$, then $X^*(\Gamma\mathcal{A})X$ and $X^*(J\mathcal{A})X$ are nonsingular.*

Proof. We only consider 2). The rest can be shown in a similar way.

'Only if': Assume that λ_0 is not semi-simple. Then by Lemma 10 in [14], there exists an eigenvector $x (= Xv$ for some $v)$ such that $y^*JEx = y^*JAx = 0$ for all eigenvectors y associated with λ_0. But then $X^*(J\mathcal{E})X$ and $X^*(J\mathcal{A})X$ are singular which is a contradiction. Hence, λ_0 is semi-simple.

'If': Let λ_0 be semi-simple. Then taking $U = X$ and $A = \lambda_0 I$, $B = I$ in Lemma 6.2, it follows by case 4 of Lemma 6.2 that $\det(X^*(J\mathcal{E})X) \neq 0$, and hence, $X^*(J\mathcal{A})X = \lambda_0 X^*(J\mathcal{E})X$ is also nonsingular. □

In the following we will reduce the pencil $\lambda\mathcal{E} - \mathcal{A}$ to an almost anti-triangular form by using unitary transformations as much as possible.

Definition 6.4 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be a regular, Γ-selfadjoint and skew-Hamiltonian/-Hamiltonian pencil. We say that $\lambda\mathcal{E} - \mathcal{A}$ is in almost anti-triangular form, if it has the form*

$$\lambda \left[\begin{array}{ccc|ccc} 0 & 0 & E_{13} & 0 & 0 & 0 \\ 0 & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{31} & E_{32} & E_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & E_{31}^* \\ 0 & 0 & 0 & 0 & E_{22}^* & E_{32}^* \\ 0 & 0 & 0 & E_{13}^* & E_{23}^* & E_{33}^* \end{array}\right] - \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & G_{13} \\ 0 & 0 & 0 & 0 & G_{22} & G_{23} \\ 0 & 0 & 0 & G_{13}^* & G_{23}^* & G_{33} \\ \hline 0 & 0 & H_{13} & 0 & 0 & 0 \\ 0 & H_{22} & H_{23} & 0 & 0 & 0 \\ H_{13}^* & H_{23}^* & H_{33} & 0 & 0 & 0 \end{array}\right], \tag{24}$$

where $E_{22}, G_{22}, H_{22} \in \mathbb{C}^{(n-m)\times(n-m)}$ are diagonal, $E_{13}, E_{31}, G_{13}, H_{13} \in \mathbb{C}^{m\times m}$ are lower anti-triangular, and m is chosen maximal.

In the following we describe a reduction method for the computation of an almost anti-triangular form. Each step of this method requires the knowledge of a single eigenvalue, an eigenvalue pair $\{\lambda_0, -\lambda_0\}$, or an eigenvalue quadruple, together with the associated deflating subspaces of a doubly structured pencil in the form (13).

Theorem 6.5 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be an ∞-regular, Γ-selfadjoint and skew-Hamiltonian/-Hamiltonian pencil.*

1. *If λ_0 is an eigenvalue that is non-real and not purely imaginary and has algebraic multiplicity r, then there exists a unitary matrix $\mathcal{P} = \mathrm{diag}(P_1, P_2) \in \mathbb{GP}_{2n}$ such that*

$$J^{-1}\mathcal{P}^* J(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} = \lambda \begin{bmatrix} \hat{E} & 0 \\ 0 & \hat{E}^* \end{bmatrix} - \begin{bmatrix} 0 & \hat{G} \\ \hat{H} & 0 \end{bmatrix},$$

where all three matrices $\hat{E}, \hat{G}, \hat{H}$ have the form

$$X = \begin{bmatrix} 0 & 0 & X_{13} \\ 0 & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix},$$

with $X_{13}, X_{31} \in \mathbb{C}^{r\times r}$ lower anti-triangular. Moreover, the spectrum of $\lambda\mathcal{E}-\mathcal{A}$ is equal to the union of $\{\lambda_0, -\lambda_0, \overline{\lambda}_0, -\overline{\lambda}_0\}$, determined (as a spectrum) by the pencil

$$\lambda \begin{bmatrix} 0 & E_{13} & 0 & 0 \\ E_{31} & E_{33} & 0 & 0 \\ 0 & 0 & 0 & E_{31}^* \\ 0 & 0 & E_{13}^* & E_{33}^* \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & G_{13} \\ 0 & 0 & G_{13}^* & G_{33} \\ 0 & H_{13} & 0 & 0 \\ H_{13}^* & H_{33} & 0 & 0 \end{bmatrix},$$

and the spectrum of the subpencil

$$\lambda \begin{bmatrix} E_{22} & 0 \\ 0 & E_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{22} \\ H_{22} & 0 \end{bmatrix}.$$

Moreover, the spectra of the two subpencils are disjoint.

2. *If λ_0 is such that $\lambda_0^2 \in \mathbb{R}\cup\{\infty\}$, then there exists a nonsingular matrix $\mathcal{P} = \mathrm{diag}(P_1, P_2) \in \mathbb{GP}_{2n}$ such that*

$$J^{-1}\mathcal{P}^* J(\lambda\mathcal{E}-\mathcal{A})\mathcal{P} = \lambda \begin{bmatrix} \hat{E} & 0 \\ 0 & \hat{E}^* \end{bmatrix} - \begin{bmatrix} 0 & \hat{G} \\ \hat{H} & 0 \end{bmatrix}$$

where all three matrices $\hat{E}, \hat{G}, \hat{H}$ have the form

$$X = \left[\begin{array}{c|cc|c} 0 & 0 & 0 & X_{14} \\ \hline 0 & X_{22} & 0 & X_{24} \\ 0 & 0 & X_{33} & X_{34} \\ \hline X_{41} & X_{42} & X_{43} & X_{44} \end{array}\right],$$

with $X_{14}, X_{41} \in \mathbb{C}^{p\times p}$ lower anti-triangular, and where $X_{22} \in \mathbb{C}^{q\times q}$ is a diagonal matrix, and $2p+q=r$. Moreover, the spectrum of $\lambda\mathcal{E}-\mathcal{A}$ is equal to the union of $\{\lambda_0, -\lambda_0\}$ which is determined (as a spectrum) by the subpencil

$$\lambda \begin{bmatrix} 0 & 0 & E_{14} & 0 & 0 & 0 \\ 0 & E_{22} & E_{24} & 0 & 0 & 0 \\ E_{41} & E_{42} & E_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{41}^* \\ 0 & 0 & 0 & 0 & E_{22}^* & E_{42}^* \\ 0 & 0 & 0 & E_{14}^* & E_{24}^* & E_{44}^* \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & G_{13} \\ 0 & 0 & 0 & 0 & G_{22} & G_{24} \\ 0 & 0 & 0 & G_{14}^* & G_{24}^* & G_{44} \\ 0 & 0 & H_{14} & 0 & 0 & 0 \\ 0 & H_{22} & H_{24} & 0 & 0 & 0 \\ H_{14}^* & H_{24}^* & H_{44} & 0 & 0 & 0 \end{bmatrix},$$

and the spectrum of the pencil

$$\lambda \begin{bmatrix} E_{33} & 0 \\ 0 & E_{33}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{33} \\ H_{33} & 0 \end{bmatrix}.$$

Moreover, the spectra of the two subpencils are disjoint.

Proof. In the following let the columns of $U = [U_1^T, U_2^T]^T$ form the basis of the deflating subspace associated with an eigenvalue $\lambda_0 \in \mathbb{C}\cup\{\infty\}$ of the pencil $\lambda\mathcal{E}-\mathcal{A}$.

1. If λ_0 is neither real nor purely imaginary, then there exists a matrix $A \in \mathbb{C}^{r\times r}$ that only has the single eigenvalue λ_0 and that satisfies $\mathcal{E}UA = \mathcal{A}U$. Without loss of generality, we may assume that A is upper triangular. By part 5. of Lemma 6.2 we have that

$$\begin{aligned} &EU_1A = GU_2, \\ &E^*U_2A = HU_1, \\ &U_2^*EU_1 = U_1^*HU_1 = U_2^*GU_2 = 0. \end{aligned} \tag{25}$$

We first show that U_1, U_2, HU_1 and GU_2 are all of full column rank. Note that $\overline{\lambda}_0$ is another eigenvalue of the pencil with algebraic multiplicity r. Let $V = [V_1^T,\ V_2^T]^T$ be a basis of the corresponding right deflating subspace, i.e., there is a matrix C only having the eigenvalue $\overline{\lambda}_0$ such that $\mathcal{E}VC = \mathcal{A}V$. By Lemma 6.1 $(\Gamma V)^*$ and $(JV)^*$ are bases of the left deflating subspaces associated with λ_0 and $-\lambda_0$, respectively, i.e., we have

$$C^*(\Gamma V)^*\mathcal{E} = (\Gamma V)^*\mathcal{A}, \quad (-C)^*(JV)^*\mathcal{E} = (JV)^*\mathcal{A}.$$

Hence we have

$$\det\Big((\Gamma V)^*\mathcal{E}U\Big) \neq 0, \quad \det\Big((\Gamma V)^*\mathcal{A}U\Big) \neq 0, \qquad (JV)^*\mathcal{E}U = (JV)^*\mathcal{A}U = 0.$$

Noting that $(\Gamma V)^*\mathcal{E}U = V_1^*E^*U_2 + V_2^*EU_1$ and $(JV)^*\mathcal{E}U = -V_1^*E^*U_2 + V_2^*EU_1$, we obtain that the matrices

$$V_1^*E^*U_2 = V_2^*EU_1 = \frac{1}{2}(\Gamma V)^*\mathcal{E}U, \qquad V_1^*HU_1 = V_2^*GU_2 = \frac{1}{2}(\Gamma V)^*\mathcal{A}U$$

are all nonsingular. Therefore, U_1, U_2 and HU_1, GU_2 must be of full column rank.

Let $L_1L_1^* = U_1^*H^2U_1$, $L_2L_2^* = U_2^*G^2U_2$ be Cholesky factorizations, see [8]. Then L_1, L_2 are lower triangular and nonsingular. Without loss of generality we may assume that both U_1, U_2 are orthonormal. By the third equation in (25), then $[U_1,\ HU_1L_1^{-*}Z_r]$ and $[U_2,\ GU_2L_2^{-*}Z_r]$ are orthonormal. Let $P_1, P_2 \in \mathbb{C}^{n\times(n-2r)}$ be orthonormal such that the columns of $[P_1^T, P_2^T]^T$ form a basis of the deflating subspace associated with all eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$ that are distinct from λ_0. Then

$$\mathcal{P}_1 = [U_1,\ P_1,\ HU_1L_1^{-*}Z_r], \qquad \mathcal{P}_2 = [U_2,\ P_2,\ GU_2L_2^{-*}Z_r],$$

are unitary. Introducing $\mathcal{P} = \operatorname{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$ and using the relations in (25) one can easily verify that

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda\begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where

$$\begin{aligned}
E_1 &= \begin{bmatrix} 0 & 0 & A^{-*}L_1Z_r \\ 0 & P_2^*EP_1 & P_2^*EHU_1L_1^{-*}Z_r \\ Z_rL_2^*A^{-1} & Z_rL_2^{-1}U_2^*GEP_1 & Z_rL_2^{-1}U_2^*GEHU_1L_1^{-*}Z_r \end{bmatrix}, \\
G_1 &= \begin{bmatrix} 0 & 0 & L_2Z_r \\ 0 & P_2^*GP_2 & P_2^*G^2U_2L_1^{-*}Z_r \\ Z_rL_2^* & Z_rL_2^{-1}U_2^*G^2P_2 & Z_rL_2^{-1}U_2^*G^3U_2L_2^{-*}Z_r \end{bmatrix}, \\
H_1 &= \begin{bmatrix} 0 & 0 & L_1Z_r \\ 0 & P_1^*HP_1 & P_1^*H^2U_1L_1^{-*}Z_r \\ Z_rL_1^* & Z_rL_1^{-1}U_1^*H^2P_1 & Z_rL_1^{-1}U_1^*H^3U_1L_1^{-*}Z_r \end{bmatrix}.
\end{aligned}$$

For example, $(U_2^*E)HU_1L_1^{-*}Z_r = (A^{-*}U_1^*H)HU_1L_1^{-*}Z_r = A^{-*}L_1Z_r$ gives the $(1,3)$-block of E_1. Note that the matrices $A^{-*}L_1Z_r$, L_2Z_r and L_1Z_r are all lower anti-triangular. The assertion about the spectrum is then easy to verify.

2. Assume that λ_0 is such that $\lambda_0^2 \in \mathbb{R} \cup \{\infty\}$. In this case we have to consider four different situations for an eigenvalue λ_0, namely, non-zero real, purely imaginary, zero and infinity.

2.1 If λ_0 is real non-zero, then there exists a matrix $A \in \mathbb{C}^{r\times r}$ having the only eigenvalue λ_0 such that $\mathcal{E}UA = \mathcal{A}U$. By part 3. of Lemma 6.2 we have

$$EU_1A = GU_2, \qquad E^*U_2A = HU_1, \qquad U^*(J\mathcal{E})U = U_1^*E^*U_2 - U_2^*E^*U_1 = 0$$

and thus, the matrices

$$\begin{aligned} T &:= U_2^*EU_1 = (U_2^*EU_1)^* = \frac{1}{2}U^*(\Gamma\mathcal{E})U, \\ & \quad\; U_1^*HU_1 = U_2^*GU_2 = A^*T = TA \end{aligned}$$

are nonsingular. Clearly then U_1, U_2, HU_1, GU_2 are of full column rank. Let $V = [V_1^T,\ V_2^T]^T \in \mathbb{C}^{2n\times s}$ form the basis of the right eigenspace of $\lambda\mathcal{E} - \mathcal{A}$ associated with λ_0, i.e., $\lambda_0\mathcal{E}V = \mathcal{A}V$. Since $\operatorname{range} V$ is a subspace of $\operatorname{range} U$, we still have V_1, HV_1, V_2, GV_2 of full column rank. Similarly, we have

$$\begin{aligned} &\lambda_0 EV_1 = GV_2, \quad \lambda_0E^*V_2 = HV_1, \\ &V_1^*HV_1 = V_2^*GV_2 = \lambda_0(V_1^*EV_2) = \lambda_0(V_2^*EV_1)^* = \frac{1}{2}\lambda_0V^*(\Gamma\mathcal{E})V =: Y, \end{aligned} \tag{26}$$

where Y is Hermitian, but possibly singular.

If Y is definite, then $V^*(\Gamma\mathcal{E})V$ is definite. By Lemma 6.3, in this case λ_0 is semi-simple, $V = U$, $A = \lambda_0 I$, and $T = U_2^*EU_1 = (U_2^*EU_1)^*$ and $U_1^*HU_1 = U_2^*GU_2 = \lambda_0T$ are both definite. Let $P_1, P_2 \in \mathbb{C}^{n\times(n-r)}$ be orthonormal such that $P_1^*E^*U_2 = P_2^*EU_1 = 0$. Then we also have $P_1^*HU_1 = P_2^*GU_2 = 0$. Let $LL^* = \delta T$ be the Cholesky factorization of $\delta T > 0$ ($\delta = \pm 1$). Introducing $\mathcal{P}_1 = [U_1L^{-*},\ P_1]$ and $\mathcal{P}_2 = [U_2L^{-*},\ P_2]$ then $\mathcal{P}_1, \mathcal{P}_2$ must be nonsingular (but in general not unitary). Indeed, if $\mathcal{P}_1$ is singular, then there exists $x = [x_1^T, x_2^T]^T \neq 0$, $x_1, x_2 \in \mathbb{C}^n$ such that $U_1L^{-*}x_1 + P_1x_2 = 0$. Pre-multiplying U_1^*H we have $U_1^*HU_1L^{-*}x_1 = 0$. Then $x_1 = 0$ and therefore $x_2 = 0$, which is a contradiction. The invertibility of $\mathcal{P}_2$ is proved in the same way. Let $\mathcal{P} = \operatorname{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$. Then we have

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where

$$E_1 = \begin{bmatrix} \delta I_r & 0 \\ 0 & P_2^*EP_1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} \lambda_0\delta I & 0 \\ 0 & P_2^*GP_2 \end{bmatrix}, \quad H_1 = \begin{bmatrix} \lambda_0\delta I & \\ 0 & P_1^*HP_1 \end{bmatrix}$$

and we have obtained the condensed form. Note that no more eigenvalues $\lambda_0, -\lambda_0$ are in the spectrum of the reduced pencil

$$\lambda \begin{bmatrix} P_2^*EP_1 & 0 \\ 0 & (P_2^*EP_1)^* \end{bmatrix} - \begin{bmatrix} 0 & P_2^*GP_2 \\ P_1^*HP_1 & 0 \end{bmatrix}.$$

If Y is not definite with inertia index $(p_1, q_1, s - p_1 - q_1)$, then assume without loss of generality $p_1 - q_1 \geq 0$. Then Y is orthogonally similar to $\operatorname{diag}(D_1, -D_2, D_3)$ where D_3 is void or scalar zero if $2p_1 \le s$ or positive diagonal of size $2p_1 - s$ if $2p_1 > s$, and D_1, D_2 are nonnegative diagonal with size $p_2 = \min\{\lfloor \frac{s}{2} \rfloor, s - p_1\}$. Using the simple fact that any 2×2 matrix $\begin{bmatrix} \delta_1 & 0 \\ 0 & -\delta_2 \end{bmatrix}$ with $\delta_1, \delta_2 \ge 0$ is orthogonally similar to $\begin{bmatrix} 0 & \sqrt{\delta_1\delta_2} \\ \sqrt{\delta_1\delta_2} & * \end{bmatrix}$, we may assume without loss of generality that $V = [V_1, V_2]$ is chosen such that V_1, V_2 are orthonormal (which will not affect the properties in (26)), and

$$Y = \begin{bmatrix} 0 & D_{12} & 0 \\ D_{12}^* & D_{22} & 0 \\ 0 & 0 & D_3 \end{bmatrix}, \tag{27}$$

where D_3 is as above, $D_{12} \in \mathbb{C}^{p_2 \times p_2}$ are nonnegative diagonal. Now partition

$$V_1 = [V_{11},\ V_{12},\ V_{13}], \quad V_2 = [V_{21},\ V_{22},\ V_{23}],$$

conformably. Obviously V_{11}, V_{21} are orthonormal and HV_{11} and GV_{21} are of full column rank. By (26) and (27) we have

$$\begin{aligned} &\lambda_0 EV_{11} = GV_{21}, \qquad \lambda_0 E^* V_{21} = HV_{11}, \\ &V_{21}^* EV_{11} = V_{11}^* HV_{11} = V_{21}^* GV_{21} = 0, \end{aligned}$$

which is the same as (25).

Similarly as in **1.** we define a unitary matrix $\mathcal{P} \in \mathbb{GP}_{2n}$ such that

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where in

$$E_1 = \begin{bmatrix} 0 & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & G_{23} \\ G_{13}^* & G_{23}^* & G_{33} \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix},$$

the matrices E_{13}, E_{13}, G_{13}, $H_{13} \in \mathbb{C}^{p_2 \times p_2}$ are all lower anti-triangular, and the matrix triple

$$\begin{bmatrix} 0 & E_{13} \\ E_{31} & E_{33} \end{bmatrix}, \quad \begin{bmatrix} 0 & G_{13} \\ G_{13}^* & G_{33} \end{bmatrix}, \quad \begin{bmatrix} 0 & H_{13} \\ H_{13}^* & H_{33} \end{bmatrix}$$

is associated with a pair of eigenvalues $\lambda_0, -\lambda_0$.

If the pencil

$$\lambda \begin{bmatrix} E_{22} & 0 \\ 0 & E_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{22} \\ H_{22} & 0 \end{bmatrix}$$

still has an eigenvalue λ_0, we can repeat the above procedure for this pencil to get a condensed form with larger anti-triangular part as before. Obviously, this procedure will finish after finitely many steps and we then have the required form.

2.2 If $\lambda_0 = i\alpha$ is purely imaginary, then there exists a matrix $A \in \mathbb{C}^{r \times r}$ having the only eigenvalue λ_0 such that $\mathcal{E}UA = \mathcal{A}U$. By 4. of Lemma 6.2 we have

$$EU_1A = GU_2, \quad E^*U_2A = HU_1$$

and the matrices

$$\begin{aligned} T &:= U_2^* E U_1 = -(U_2^* E U_1)^* = \frac{1}{2} U^* (J\mathcal{E}) U, \\ U_1^* H U_1 &= -U_2^* G U_2 = -A^* T = TA \end{aligned}$$

are nonsingular. Replacing T by iT which is Hermitian, and A by $-iA$ which has the real eigenvalue α we can use the same proof as in **2.1**.

2.3 For $\lambda_0 = 0$, there exists a matrix $A \in \mathbb{C}^{2r \times 2r}$ having the only eigenvalue zero such that $\mathcal{E} U A = \mathcal{A} U$. Here the number of columns of U must be even by the canonical form. By 2. of Lemma 6.2 we have

$$\begin{aligned} & EU_1 A = GU_2, \quad E^* U_2 A = HU_1, \\ & T := U_2^* E U_1, \quad \det U^* (\Gamma \mathcal{E}) U = \det (T + T^*) \neq 0, \quad \det U^* (J\mathcal{E}) U = \det (T - T^*) \neq 0, \\ & U_1^* H U_1 = A^* T = T^* A, \quad U_2^* G U_2 = A^* T^* = TA. \end{aligned}$$

As before, let $V = [V_1^T \; V_2^T]^T$ be a basis of the right eigenspace of $\lambda \mathcal{E} - \mathcal{A}$, i.e., $\mathcal{E} V$ is of full column rank and $\mathcal{A} V = 0$. Without loss of generality we assume that $V_1 = [V_{11}, \; 0]$ and V_{11} is of full column rank, which can be obtained by performing an LQ decomposition of V_1. Partition $V_2 = [V_{21}, V_{22}]$ conformably. Then V_{22} must be of full column rank, since ΣV is another basis of the right eigenspace. From the uniqueness of the eigenspace it follows that there exists a nonsingular matrix F such that

$$V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} V_{11} & 0 \\ -V_{21} & -V_{22} \end{bmatrix} F,$$

and one has

$$F = \begin{bmatrix} I & 0 \\ -2F_1 & -I \end{bmatrix}, \qquad V_{21} = V_{22} F_1.$$

So we may assume further that V is already in the form

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix},$$

where $V_1 \in \mathbb{C}^{n \times p_1}$ nd $V_2 \in \mathbb{C}^{n \times p_2}$. Moreover, we have that EV_1 and $E^* V_2$ are of full column rank and $GV_2 = 0$, $HV_1 = 0$.

We then consider one step of reduction in the following subcases.

Subcase 1. $p_1, p_2 > 0$ and at least one, say p_1, is larger than 1. (The case $p_2 > 1$ can be treated analogously.) Then $V_2^* E V_1$ is not void and the number of columns is $p_1 > 1$. By applying a permuted QR-factorization, $V_2^* E V_1$ can be reduced to a form $\begin{bmatrix} 0 \\ R \end{bmatrix}$ when $p_2 > p_1$ or $[0, \; R]$ when $p_2 \le p_1$, where R is square and lower anti-triangular. From this condensed form we obtain full rank matrices X_1, X_2 with the same number of columns such that

$$X_2^* V_2^* E V_1 X_1 = 0. \tag{28}$$

We still have

$$HV_1 X_1 = GV_2 X_2 = 0 \tag{29}$$

and EV_1X_1, $E^*V_2X_2$ are of full column rank. Let

$$\mathcal{P}_1 = [V_1X_1,\ P_1,\ E^*V_2X_2], \qquad \mathcal{P}_2 = [V_2X_2,\ P_2\ EV_1X_1]$$

be square, where P_1, P_2 are chosen such that

$$P_1^*[V_1X_1,\ E^*V_2X_2] = 0, \quad P_2^*[V_2X_2,\ EV_1X_1] = 0. \tag{30}$$

Then (28) implies that $\mathcal{P}_1$, $\mathcal{P}_2$ are nonsingular and we have $\mathcal{P} = \mathrm{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$. Then

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where by (28)–(30) we have

$$\begin{aligned} E_1 &= \mathcal{P}_2^* E \mathcal{P}_1 = \begin{bmatrix} 0 & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}, \\ G_1 &= \mathcal{P}_2^* G \mathcal{P}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & G_{22} & G_{23} \\ 0 & G_{23}^* & G_{33} \end{bmatrix}, \quad H_1 = \mathcal{P}_1^* H \mathcal{P}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & H_{22} & H_{23} \\ 0 & H_{23}^* & H_{33} \end{bmatrix}, \end{aligned}$$

with $E_{13} = X_2^*V_2^*EE^*V_2X_2$, $E_{31} = X_1^*V_1^*E^*EV_1X_1$. Applying an additional transformation we can reduce E_{13} and E_{31} to lower anti-triangular form. From the process, we see that the transformation matrix can be chosen unitary.

Subcase 2. If $p_1 = p_2 = 1$, then $V^*\Gamma\mathcal{E}V = \begin{bmatrix} 0 & V_1^*E^*V_2 \\ V_2^*EV_1 & 0 \end{bmatrix}$ is 2×2. If $V_2^*EV_1 = 0$, then one can apply the reduction of Subcase 1. If $V_2^*EV_1 \neq 0$ then $\det(V^*\Gamma\mathcal{E}V) \neq 0$. Since $V^*\Gamma$ is a basis of the left eigenspace, by Lemma 6.3 the eigenvalue 0 is semi-simple and the algebraic multiplicity is 2, and hence V is just a basis of the right deflating subspace. Let

$$\mathcal{P}_1 = [V_1,\ P_1], \qquad \mathcal{P}_2 = [V_2,\ P_2],$$

be square, where P_1, P_2 satisfy $V_2^*EP_1 = 0$ and $P_2^*EV_1 = 0$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are nonsingular. Indeed, if there exists a scalar α and a vector x such that $V_1\alpha + P_1x = 0$, then pre-multiplying by V_2^*E one gets $V_2^*EV_1\alpha = 0$, which implies $\alpha = 0$ and hence $x = 0$. So $\det \mathcal{P}_1 \neq 0$. In the same way one obtains $\det \mathcal{P}_2 \neq 0$. With $\mathcal{P} = \mathrm{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$, we obtain

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where

$$\begin{aligned} E_1 &= \mathcal{P}_2^* E \mathcal{P}_1 = \begin{bmatrix} V_2^*EV_1 & 0 \\ 0 & E_{22} \end{bmatrix}, \\ G_1 &= \mathcal{P}_2^* G \mathcal{P}_2 = \begin{bmatrix} 0 & 0 \\ 0 & G_{22} \end{bmatrix}, \qquad H_1 = \mathcal{P}_1^* H \mathcal{P}_1 = \begin{bmatrix} 0 & 0 \\ 0 & H_{22} \end{bmatrix}. \end{aligned}$$

Again here $\mathcal{P}$ can be chosen unitary. Note that H_{22} and G_{22} cannot be singular in this case and no more zero eigenvalue occurs in the reduced pencil

$$\lambda \begin{bmatrix} E_{22} & 0 \\ 0 & E_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{22} \\ H_{22} & 0 \end{bmatrix}.$$

Subcase 3. If $p_2 = 0$ (or $p_1 = 0$ which can be treated analogously), then we have $V = [V_1^T,\ 0]^T$, EV_1 is of full column rank and $HV_1 = 0$. Moreover, G must be nonsingular, since otherwise there would be additional eigenvectors as $[0,\ x_2^T]^T$ with $x_2 \neq 0$ associated with a zero eigenvalue of G and $p_2 > 0$. Let V_2 satisfy

$$GV_2 = EV_1. \tag{31}$$

Then $V_2 \in \mathbb{C}^{n\times p_1}$ is of full column rank. If $V_2^* G V_2$ is not definite, then one can determine a full rank matrix X such that $X^* V_2^* G V_2 X = 0$. Then

$$X^* V_2^* E V_1 X = X^* V_2^* G V_2 X = 0.$$

Clearly $HV_1X = 0$ and $V_1X, V_2X, EV_1X = GV_2X$ are of full column rank. With these properties one can determine nonsingular matrices

$$\mathcal{P}_1 = [V_1X,\ P_1,\ E^*V_2X], \qquad \mathcal{P}_2 = [V_2X,\ P_2,\ EV_1X],$$

where $P_1^*[V_1X,\ E^*V_2X] = P_2^*[V_2X,\ EV_1X] = 0$ and $\mathcal{P}_1, \mathcal{P}_2$ can be chosen to be unitary. Then $\mathcal{P} = \operatorname{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where

$$\begin{aligned} E_1 &= \mathcal{P}_2^* E \mathcal{P}_1 = \begin{bmatrix} 0 & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}, \\ G_1 &= \mathcal{P}_2^* G \mathcal{P}_2 = \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & G_{23} \\ G_{13}^* & G_{23}^* & G_{33} \end{bmatrix}, \qquad H_1 = \mathcal{P}_1^* H \mathcal{P}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & H_{22} & H_{23} \\ 0 & H_{23}^* & H_{33} \end{bmatrix}, \end{aligned}$$

and $E_{13} = X^*V_2^*EE^*V_2X$, $E_{31} = X^*V_1^*E^*EV_1X$, $G_{13} = X^*V_2G^2V_2X$. Let Q_1, Q_2, Q_3 be unitary such that $Q_1^*E_{13}$, $\quad Q_1^*G_{13}Q_2$, $\quad Q_2^*E_{31}Q_3$ are all lower anti-triangular, which can be done by performing QR-like factorizations to E_{13} first to determine Q_1 such that $Q_1^*E_{13}$ is lower anti-triangular, then to $Q_1^*G_{13}$ to determine Q_2 and finally to $Q_2^*E_{31}$ to determine Q_3. Set $\mathcal{Q}_1 = \operatorname{diag}(Q_3, I, I)$, $\mathcal{Q}_2 = \operatorname{diag}(Q_1, I, Q_2)$ and $\mathcal{Q} = \operatorname{diag}(\mathcal{Q}_1, \mathcal{Q}_2) \in \mathbb{GP}_{2n}$. Then $(J^{-1}\mathcal{Q}J)(\lambda\mathcal{E}_1 - \mathcal{A}_1)\mathcal{Q}$ has the desired form.

If $V_2^*GV_2$ is definite then $V_2^*EV_1 = V_2^*GV_2$ is also definite. Set

$$\mathcal{P}_1 = [V_1(\delta V_2^*GV_2)^{-\frac{1}{2}},\ P_1], \qquad \mathcal{P}_2 = [V_2(\delta V_2^*GV_2)^{-\frac{1}{2}},\ P_2],$$

where $\delta \in \{1, -1\}$ is such that $\delta V_2^*GV_2 > 0$ and P_1, P_2 satisfy $V_2^*EP_1 = 0$ and $P_2^*EV_1 = 0$. Similarly as before we see that $\mathcal{P}_1$ and $\mathcal{P}_2$ are nonsingular. With $\mathcal{P} = \operatorname{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$, then

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where

$$E_1 = \begin{bmatrix} \delta I_{p_1} & 0 \\ 0 & E_{22} \end{bmatrix}, \qquad G_1 = \begin{bmatrix} \delta I_{p_1} & 0 \\ 0 & G_{22} \end{bmatrix}, \qquad H_1 = \begin{bmatrix} 0_{p_1} & 0 \\ 0 & H_{22} \end{bmatrix}.$$

Since H_{22} and G_{22} must be nonsingular (V_1 is also a basis of the null space of H with dimension p_1), no more zero eigenvalue is in the reduced pencil

$$\lambda \begin{bmatrix} E_{22} & 0 \\ 0 & E_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{22} \\ H_{22} & 0 \end{bmatrix}.$$

If after one step of this reduction the pencil

$$\lambda \begin{bmatrix} E_{22} & 0 \\ 0 & E_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{22} \\ H_{22} & 0 \end{bmatrix}$$

still has a zero eigenvalue, we repeat the procedure and obtain the desired form after finitely many steps.

2.4 If $\lambda_0 = \infty$, then there exists a matrix $B \in \mathbb{C}^{2r\times 2r}$ having the only eigenvalue zero such that $\mathcal{E}U = \mathcal{A}UB$. Here the number of columns of U must be even, since we have assumed that the pencil is ∞-regular. By 1. of Lemma 6.2 we have

$$\begin{aligned} &EU_1 = GU_2B, \quad E^*U_2 = HU_1B, \\ &T_1 := U_1^*HU_1, \quad T_2 = U_2^*GU_2, \quad \det(T_1 \pm T_2) \neq 0, \\ &U_2^*EU_1 = B^*T_1 = T_2B. \end{aligned}$$

Again let $V = [V_1^T\ V_2^T]^T$ be a basis of the right eigenspace of $\lambda\mathcal{E} - \mathcal{A}$, i.e., $\mathcal{E}V = 0$ and $\mathcal{A}V$ is of full column rank. Using the fact that ΣV is also a basis of the right eigenvector subspace as in **2.3** we assume $V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$, where $V_1, V_2 \in \mathbb{C}^{n\times p_1}$ are of full column rank. It is clear that V_1 and V_2 have the same number of columns, since V_1 and V_2 span the null space of E and E^* respectively. We have that $EV_1 = E^*V_2 = 0$ and GV_2 and HV_1 are of full column rank. Consider the matrices $V_2^*GV_2$ and $V_1^*HV_1$ and the following two subcases.

Subcase 1. If both matrices are indefinite, then there exits matrices Z_1, Z_2 such that $Z_2^*V_2^*GV_2Z_2 = 0$, $Z_1^*V_1^*HV_1Z_1 = 0$. Obviously we can choose Z_1, Z_2 such that they have the same number of columns. If originally Z_2 has more columns than Z_1, then we just choose a submatrix of Z_2 to be a new Z_2 which has the same number of columns as Z_1. We then can determine two nonsingular matrices

$$\mathcal{P}_1 = [V_1Z_1,\ P_1, HV_1Z_1], \quad \mathcal{P}_2 = [V_2Z_2,\ P_2\ GV_2Z_2]$$

such that $P_1^*[V_1Z_1,\ HV_1Z_1] = P_2^*[V_2Z_2,\ GV_2Z_2] = 0$. With $\mathcal{P} = \mathrm{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$, then

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & E_{33} \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & G_{23} \\ G_{13}^* & G_{23}^* & G_{33} \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix}.$$

Again, in this subcase $\mathcal{P}$ can be chosen unitary.

Subcase 2. Suppose that one of the matrices $V_1^*HV_1$, $V_2^*GV_2$ is definite, say, without loss of generality, $V_1^*HV_1$. If $V_2^*GV_2$ is nonsingular then there exist nonsingular matrices X_1, X_2

such that $X_1^* V_1^* H V_1 X_1 = \delta I$ and $X_2^* V_2^* G V_2 X_2 = \Theta$, where $\delta \in \{1, -1\}$ and Θ is a signature matrix. Defining square matrices

$$\mathcal{P}_1 = [V_1 X_1,\ P_1], \qquad \mathcal{P}_2 = [V_2 X_2,\ P_2]$$

such that P_1 and P_2 have full rank and satisfy $P_1^* H V_1 X_1 = 0$ and $P_2^* G V_2 X_2 = 0$ and setting $\mathcal{P} = \mathrm{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$, it is easy to verify that $\mathcal{P}$ is nonsingular. We obtain

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & E_{22} \end{bmatrix}, \quad G_1 = \begin{bmatrix} \Theta & 0 \\ 0 & G_{22} \end{bmatrix}, \qquad H_1 = \begin{bmatrix} \delta I & 0 \\ 0 & H_{22} \end{bmatrix}.$$

Since E_{22} must be nonsingular, no more infinite eigenvalue is in the reduced pencil

$$\lambda \begin{bmatrix} E_{22} & 0 \\ 0 & E_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{22} \\ H_{22} & 0 \end{bmatrix}.$$

If $V_2^* G V_2$ were singular, then as above there would exist X_1, X_2 nonsingular such that $X_1^* V_1^* H V_1 X_1 = \delta I$ and $X_2^* V_2^* G V_2 X_2 = \mathrm{diag}(0, \Theta)$. Let $X_2 = [X_{12}, X_{22}]$ be such that $X_{12}^* V_2^* G V_2 X_{12} = 0$ and let

$$\mathcal{P}_1 = [V_1 X_1,\ P_1], \qquad \mathcal{P}_2 = [V_2 X_2,\ P_2]$$

be square, where P_1, P_2 are of full rank and satisfy $P_1^* H V_1 X_1 = 0$ and $P_2^* [V_2 X_{12},\ G V_2 X_{22}] = 0$. Then one can verify that $\mathcal{P}_1$, $\mathcal{P}_2$ are nonsingular. With $\mathcal{P} = \mathrm{diag}(\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{GP}_{2n}$, then

$$(J^{-1}\mathcal{P}J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{P} =: \lambda\mathcal{E}_1 - \mathcal{A}_1 := \lambda \begin{bmatrix} E_1 & 0 \\ 0 & E_1^* \end{bmatrix} - \begin{bmatrix} 0 & G_1 \\ H_1 & 0 \end{bmatrix},$$

where

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & E_{22} \end{bmatrix}, \quad G_1 = \begin{bmatrix} \hat{\Theta} & 0 \\ 0 & G_{22} \end{bmatrix}, \qquad H_1 = \begin{bmatrix} \delta I & 0 \\ 0 & H_{22} \end{bmatrix},$$

with $\hat{\Theta} = \mathrm{diag}(0, \Theta)$. Then it is obvious that $\lambda\mathcal{E}_1 - \mathcal{A}_1$ would be a singular pencil. Hence $V_2^* G V_2$ must be invertible.

If the subpencil

$$\lambda \begin{bmatrix} E_{22} & 0 \\ 0 & E_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{22} \\ H_{22} & 0 \end{bmatrix}$$

still has infinite eigenvalues, then we repeat the procedure, so that after finitely many steps we obtain the desired form. □

Remark 6.6 Theorem 6.5 gives rise to a step-by-step reduction procedure in which we continue for every eigenvalue with the pencil

$$\lambda \begin{bmatrix} E_{22} & 0 \\ 0 & E_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{22} \\ H_{22} & 0 \end{bmatrix}$$

after case 1 or

$$\lambda \begin{bmatrix} E_{33} & 0 \\ 0 & E_{33}^* \end{bmatrix} - \begin{bmatrix} 0 & G_{33} \\ H_{33} & 0 \end{bmatrix}$$

after each subcase of case 2. In this way we can get the almost anti-triangular form. This is because in each subcase of case 2 we do not get the anti-triangular form except for the last step, where the corresponding block associated with either H or G (or both) is definite.

Note that the non-unitary transformations may have to be performed in the final step of four subcases of case 2 only. These transformations can be carried out even after all possible unitary transformations for all eigenvalues having been performed. Moreover, the non-unitary transformations can be performed in a robust way because of the definiteness of one or both of the blocks related to H and G.

Note that the eigenvector reduction procedure used in case 2 can also be used in case 1. Then in each step of the reduction one only has to determine the eigenspaces.

7 Conclusion

We have presented canonical forms for double structured matrices and pencils and then given necessary and sufficient conditions when analogous condensed forms can be determined via unitary transformations. In these cases we expect to be able to construct these forms via numerically stable structure preserving algorithms. If this is not possible, then we can construct almost anti-triangular forms also using non-unitary transformations

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Appendix

For the case of matrix pencils that are not ∞-regular we can also design a canonical form. We state this result here for completeness.

Theorem 7.1 *Let $\lambda\mathcal{E} - \mathcal{A} \in \mathbb{C}^{2n\times 2n}$ be a regular, Γ-selfadjoint, and skew-Hamiltonian/-Hamiltonian pencil. Then there exists a nonsingular matrix $\mathcal{W} \in \mathbb{GP}_{2n}$ such that*

$$(\Gamma^{-1}\mathcal{W}^*\Gamma)(\lambda\mathcal{E} - \mathcal{A})\mathcal{W} \quad = \quad (J^{-1}\mathcal{W}^*J)(\lambda\mathcal{E} - \mathcal{A})\mathcal{W}$$

$$= \lambda \begin{bmatrix} I_{n_f} & 0 & 0 & 0 \\ 0 & E_\infty & 0 & 0 \\ 0 & 0 & I_{n_f} & 0 \\ 0 & 0 & 0 & E_\infty^* \end{bmatrix} - \begin{bmatrix} 0 & 0 & G_f & 0 \\ 0 & 0 & 0 & G_\infty \\ H_f & 0 & 0 & 0 \\ 0 & H_\infty & 0 & 0 \end{bmatrix},$$

where G_f and H_f in the canonical form (8) of Theorem 3.2 and

$$E_\infty = \mathrm{diag}(E_1, \dots, E_k), \; G_\infty = \mathrm{diag}(G_1, \dots, G_k), \; H_\infty = \mathrm{diag}(H_1, \dots, H_k),$$

and the blocks E_j, G_j, and H_j have corresponding sizes and are of one and only one of the following forms:

1. *blocks corresponding to paired even sized blocks in type 4.1.1 of Theorem 4.1 with sizes $2p$, associated with the eigenvalue ∞:*

$$E_j = Z_{2p}\mathcal{J}_{2p}(0) \quad \text{and} \quad G_j = H_j = Z_{2p};$$

2. *blocks corresponding to two odd sized blocks in type 4.1.2 of Theorem 4.1 associated with the eigenvalue ∞ with sizes $2p+1$, $2q+1$ and $p \geq q$, and the structure indices ε_1, δ_1 and ε_2, δ_2 chosen such that $\varepsilon_1\delta_1 = -\varepsilon_2\delta_2$:*

$$E_j = \left[\begin{array}{c|c} 0 & \varepsilon_1 I_p \\ \hline \begin{matrix} 0 & \\ & \varepsilon_2 I_q \end{matrix} & 0 \end{array}\right], \quad H_j = \begin{bmatrix} \varepsilon_1 Z_{p+1} & 0 \\ 0 & \varepsilon_2 Z_q \end{bmatrix}, \quad G_j = \begin{bmatrix} \varepsilon_2 Z_{q+1} & 0 \\ 0 & \varepsilon_1 Z_p \end{bmatrix}$$

when $\varepsilon_1\delta_1 = 1$, or

$$E_j = \left[\begin{array}{c|c} 0 & \begin{matrix} 0 & \\ & \varepsilon_2 I_q \end{matrix} \\ \hline \varepsilon_1 I_p & 0 \end{array}\right], \quad H_j = \begin{bmatrix} \varepsilon_2 Z_{q+1} & 0 \\ 0 & \varepsilon_1 Z_p \end{bmatrix}, \quad G_j = \begin{bmatrix} \varepsilon_1 Z_{p+1} & 0 \\ 0 & \varepsilon_2 Z_q \end{bmatrix}$$

when $\varepsilon_1\delta_1 = -1$.

Proof. The proof is extremely technical and not presented here. □